THREE TRICKS FOR THE KPZ EQUATION

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Abstract. These lecture notes were prepared for the Summer School on Topics at the Interface of PDE and Probability at Carnegie Mellon University and the Summer School on PDEs and Randomness at the Max Planck Institute in 2026. They present a stochastic-analytic perspective on random fluctuations and invariant measures for the 1d Kardar–Parisi–Zhang equation. The results discussed here are based on joint works with Alex Dunlap, Tomasz Komorowski, Jeremy Quastel, and Tommaso Rosati [12, 13, 17].

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1. Lecture 1: Introduction

This is a work in progress and in particular there are many missing references.

The goal of these lectures is to present some stochastic analytic approaches in the study of the KPZ equation, which aims at understanding the 1:2:3 scaling in the 1+1 KPZ universality class. The equation we are interested in was written down 40 years ago and takes the form

$$\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \xi. \tag{1.1}$$

It was considered as one of the default models of surface growth subjected to random perturbations. Throughout most of the lectures, ξ is a spacetime white noise, i.e., the generalized Gaussian process with zero mean and covariance function given by

$$\mathbf{E}\,\xi(t,x)\xi(s,y) = \delta(t-s)R(x-y), \qquad R(\cdot) = \delta(\cdot).$$

Some developed tools applies to the spatially colored case with $R \in C_c^\infty$, which is actually the case of interest from my perspective. The equation is posed on the whole line, and we will also discuss the compact setting with periodic or Neumann boundary conditions.

Without the nonlinear term, (1.1) becomes the so-called Edwards-Wilkinson equation $\partial_t U = \frac{1}{2}\Delta U + \xi$, and it is straightforward to check using the scaling property of ξ that this linear equation is $1:2:4$ scaling invariant: $\varepsilon U(\frac{t}{\varepsilon^4}, \frac{x}{\varepsilon^2})$ solves the same equation as U. The solution is a Gaussian process for which everything can be computed, and it serve as the limit in many central limit type results. With the additional nonlinear term, the problem is much more difficult, and the goal is to understand the large scale behaviors of (1.1), in particular how the 1:2:4 scaling is changed.

The answer to the question is known, at least for the case of $R(\cdot) = \delta(\cdot)$. Instead of considering the 1:2:4 scaling for the Edwards-Wilkinson equation, one should scale (1.1) by 1:2:3 and consider $\varepsilon h(\frac{t}{\varepsilon^3}, \frac{x}{\varepsilon^2})$. The limiting object is the so-called KPZ fixed point, a scaling invariant Markov process constructed in [21]. In the last quarter century, centering around understanding the scaling limit of one dimensional random growth models, tremendous progresses has been made, and nowadays there are many solvable models for which one can prove such results.

Most of the results for (1.1) is derived through a discretization, which means one finds an appropriate discrete model which approximates (1.1) in certain regimes and by extracting information from the discrete model one draws conclusions on (1.1). Up to today, this is the most effective approach, in the sense that almost all the important information on the limit of (1.1) was obtained in this way.

The goal of these lectures is to provide different perspectives to study equations like (1.1). Although the developed approaches do not lead to as strong results as determining the limit, it offers different ways of thinking about the problem which in some cases lead to new applications.

In the following sections, I sketch the main ideas in these lectures. The "three tricks" refer to (i) the Clark-Ocone formula (ii) Gaussian integration by parts (iii) time reversal.

1.1. Infra-red cutoff and homogenization.

The first perspective we took is to periodize the problem, prove a homogenization type result, and then try to understand how homogenization breaks down as we remove the periodization. Instead of considering (1.1) on $\mathbb{R}$, we pose it on the circle $\mathbb{T}_L$, with the length L fixed.

This removes the low frequencies/long wavelengths in consideration and provides the problem with more mixing so that it can be homogenized in the usual way.

This type of ideas was used in the study of anomalous diffusion in random media. For example, in order to study the super-diffusive behaviors in turbulent transport, it is natural to analyze how the effective diffusivity diverges as the domain in consideration becomes larger and larger. In [14], the so-called box diffusivity was defined, and its asymptotics was analyzed to derive the relevant scaling exponents of several models. More recently, for the model of a Brownian particle in the curl of a two-dimensional Gaussian free field, the same approach was adopted in [1, 7], and through a "progressive" homogenization, the precise order of the mean square displacement and a quenched central limit theorem were derived.

Suppose (1.1) is posed on $\mathbb{T}_L$ and we consider

$$\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \xi, \qquad t > 0, x \in \mathbb{T}_L. \tag{1.2}$$

Since Δ on $\mathbb{T}_L$ has a spectral gap, we expect that the Markov process associated with (1.2) mixes faster than the one associated with (1.1). As a matter of fact, it is not hard to convince oneself that the mixing is exponentially fast thanks to the spectral gap. Due to the fast mixing, one may expect a central limit type result. Indeed, we showed that

Theorem 1.1. *Consider $R(\cdot) = \delta(\cdot)$ or $R \in C_c^\infty$. Take any $x \in \mathbb{T}_L$ and any initial data for (1.2),*

$$\frac{h(t,x) + \gamma_L t}{\sqrt{t}} \Rightarrow N(0, \sigma_L^2)$$

as $t \to \infty$. Here γ_L, σ_L^2 are two positive constants.

The result shows the fluctuation of h at time t is $t^{1/2}$ which is $\gg t^{1/3}$. It does not say anything about the spatial correlation or the convergence of $h(t,x)$ as a process indexed by x, which is partly due to the fact that in sending $t \to \infty$ in Theorem 1.1, the length L is fixed. Still, one can talk about the convergence as a process, and here is the result in this regard: define the relative height

$$\tilde{h}(t,x) = h(t,x) - h(t,0) \tag{1.3}$$

Proposition 1.2. *Consider $R(\cdot) = \delta(\cdot)$ or $R \in C_c^\infty$. As $t \to \infty$, $\tilde{h}(t,\cdot)$ converges in distribution in $C(\mathbb{T}_L)$. In the case of $R(\cdot) = \delta(\cdot)$, the limiting law is the Brownian bridge on $\mathbb{T}_L$ satisfying $B(0) = B(L) = 0$.*

To understand the above result, we first note that $\tilde{h}$ contains the same information as $u(t,x) = \nabla h(t,x)$. The u solves the stochastic Burgers equation

$$\partial_t u = \tfrac{1}{2}\Delta u + \tfrac{1}{2}\nabla u^2 + \nabla \xi, \tag{1.4}$$

where the forcing is conservative. Due to the growth of the interface, it is $\tilde{h}$ (equivalently u) rather than h that has an invariant measure. The above proposition essentially says that $\tilde{h}$ (equivalently u) converges in distribution to the invariant measure, and in the special case of $R(\cdot) = \delta(\cdot)$, the invariant measure for $\tilde{h}$ is the Brownian bridge measure.

Combine Theorem 1.1 and Proposition 1.2, we have a more or less complete understanding of the long time behavior of (1.2). The surface $\{h(t,x)\}_{x \in \mathbb{T}_L}$ moves as a whole, with the transversal profile, which is described by $\tilde{h}$, converges to some steady state, and the "average" height fluctuates on the scale of $t^{1/2}$ and satisfies a central limit theorem. By average height, we mean either $\frac{1}{L}\int_0^L h(t,x)dx$ or $h(t,x)$ with any $x \in \mathbb{T}_L$, because according to Proposition 1.2, the difference is order $O(1)$.

The CLT result in Theorem 1.1 may be viewed as a homogenization type result with the variance σ_L^2 playing the role of homogenization constant. Comparing the whole space $t^{1/3}$ fluctuation with the periodic $t^{1/2}$ fluctuation, we expect σ_L^2 to vanish as $L \to \infty$. As a matter of fact, the decaying rate of σ_L^2 turns out to be an important quantity that is intimately related to the $1:2:3$ scaling. One may compare with the following example of diffusion in random environment: suppose a process X_t is super-diffusive with some exponent $\alpha > \frac{1}{2}$ to be determined: $X_t \sim t^\alpha$, we compute the box-diffusivity which is the diffusion constant of $X_L(t)$ where X_L is some $L-$periodic approximation of X, e.g., obtained by periodizing the random environment, denote it by σ_L^2 so that $X_L(t) \sim \sigma_L \sqrt{t}$. If the homogenization theory allows to derive $\sigma_L \sim L^\beta$ for some $\beta > 0$, then for those L such that $\sigma_L \sqrt{t} \gg L$, one still expects the homogenization behavior as the process visits many cells. This indicates that the critical length scale is given by $\sigma_L \sqrt{t} \sim L^\beta \sqrt{t} \sim L$, which leads to $L \sim t^{\frac{1}{2(1-\beta)}}$ and $X_L(t) \sim \sigma_L \sqrt{t} \sim t^{\frac{1}{2(1-\beta)}}$, i.e., $\alpha = \frac{1}{2(1-\beta)}$. This is the heuristic physical approach to relate the super-diffusive exponent with the blowing-up rate of the effective diffusivity.

To see how the above discussion applies to the periodic KPZ problem, let us combine Theorem 1.1 and Proposition 1.2 to compute the correlation between the height function at different points. After all, the $2/3$ exponent from the $1:2:3$ scaling is about the correlation length of the height function: in the whole line setting, $h(t,x)$ and $h(t,y)$ are critically correlated if $x - y \sim t^{2/3}$.

We write $h(t,x) = h(t,0) + (h(t,x) - h(t,0))$. By Proposition 1.2, we know that the transversal fluctuation $h(t,x) - h(t,0)$ converges to some steady state, which for the moment we denote by $H(x)$. Meanwhile $h(t,0)$ satisfies a CLT with variance σ_L^2, so

$$\mathrm{Var}\, h(t,0) \sim \sigma_L^2 t, \qquad \mathrm{Var}[h(t,x) - h(t,0)] \sim \mathrm{Var} H(x), \qquad \text{for } t \gg 1$$

For fixed L, we obviously have $\mathrm{Var}[h(t,x) - h(t,0)] \ll \mathrm{Var} h(t,0)$, which implies that $h(t,x)$ and $h(t,0)$ are almost perfectly correlated – this is what we mean by "the surface moves as a whole". With this in mind, we increase L, then one may expect that the transversal fluctuations to grow as $L \to \infty$. After all if the surface is rough, the transversal fluctuations are large. In the general case, however, we know very little about H, e.g. if $R(\cdot) \in C_c^\infty$, we basically only know about the existence (and uniqueness). It is completely unclear how fast $\mathrm{Var} H(x)$ grows as $x \to \infty$. Here comes the first special feature of the white noise: in the case of $R(\cdot) = \delta(\cdot)$, the law of H is *explicit* and coincide with that of a Brownian bridge on $[0, L]$ with $H(0) = H(L) = 0$. As a result, $\mathrm{Var} H(x) = x(L-x)/L$, and the largest transversal fluctuation is attained at the midpoint $x = L/2$ in which case $\mathrm{Var} H(x) = L/4$. In other words, for this special model, the transversal fluctuation is of order L. Suppose we know that $\sigma_L^2 \sim L^{-\alpha}$ for some $\alpha > 0$, then as we gradually increase L, nontrivial correlation between $h(t,0)$ and $h(t,x)$ starts appearing only when $\sigma_L^2 t \sim L$, i.e., $t \sim L^{1+\alpha}$ which should lead to the critical scale.

The above discussion can be made rigorous, and here is the main result

Theorem 1.3. *Assume ξ is spacetime white noise, we have the effective diffusivity takes the form*

$$\sigma_L^2 = \mathbf{E} \frac{\int_0^L e^{(W_1(y) + W_2(y) + 2W_3(y))} dy}{\int_0^L e^{(W_1(y) + W_3(y))} dy \int_0^L e^{(W_2(y) + W_3(y))} dy} \sim \frac{1}{\sqrt{L}}, \qquad \text{as } L \to \infty, \tag{1.5}$$

where W_i are independent Brownian bridges on $[0, L]$. In addition, for the solution to (1.2), if $h(0,x)$ is a Brownian bridge (start at equilibrium) and $L = t^\alpha$, then

$\mathrm{Var}\, h(t,0) \sim t^{1-\frac{\alpha}{2}}$ *for* $\alpha \in [0,2/3)$. *There exists* $\delta > 0$ *such that if* $L = Ct^{2/3}$ *for* $C < \delta$, *then* $\mathrm{Var} h(t,0) \sim t^{2/3}$.

We are only able to prove the above result for spacetime white noise, but we believe it's true in the general case of $R(\cdot) \in C_c^\infty$ that $\sigma_L^2 \sim 1/\sqrt{L}$. In some sense one should view this decaying rate of diffusion constant, together with the diffusive behavior of H, as an equivalent way of describing the 1:2:3 scaling. What we mean by this is, suppose one can show in the colored noise case that (i) $\mathrm{Var} H(x) \sim x$ for $1 \ll x \ll L$ and (ii) $\sigma_L^2 \sim 1/\sqrt{L}$, then the above theorem holds.

A somewhat surprising feature of Theorem 1.3 is that the effective diffusivity σ_L^2 takes a rather explicit form. Perhaps what is more surprising is that a similar formula holds for colored noise and in high dimensions:

$$\sigma_L^2 = \int_{\mathbb{T}^{2d}} \mathbf{E}\left[\frac{e^{H_1(y_1)+H_2(y_1)+H_3(y_2)+H_2(y_2)}}{\int_{\mathbb{T}^d} e^{H_1(y')+H_2(y')} dy' \int_{\mathbb{T}^d} e^{H_3(y')+H_2(y')} dy'} \right] R(y_1 - y_2) dy_1 dy_2, \tag{1.6}$$

where H_1, H_2, H_3 are independent copies of the steady state.

In homogenization theory, it is rarely the case that the homogenization constant can be computed explicitly (sometimes in $d = 1$ maybe, but as it will become clear later, our problem is really $d = \infty$ by analogy). The fact that the effective diffusivities in (1.5) and (1.6) depend explicitly on the underlying steady state is a highly nontrivial fact and it relies crucially on the special structure of the problem.

To derive formulas in (1.5) and (1.6), the standard homogenization theory does not work well, which only expresses the effective diffusivities in terms of the so-called corrector function. It would be wonderful if one can come up with a PDE approach in the spirit of scale-by-scale homogenization to show directly that $\sigma_L^2 \sim 1/\sqrt{L}$, without appealing to the explicit formula. For the model of a diffusion in the curl of 2d GFF, we refer to [1, 7] for such discussions.

For our specific problem, the explicit formula were derived by the so-called Clark-Ocone formula, and this is the first "trick" in these lectures and will be discussed in details in Section 2.

1.2. **Integration by parts and invariant measure.** In the previous discussion, we already noted that the key advantage of working with spacetime white noise is that the invariant measure is explicit. Indeed, for (1.2), the Brownian bridge is the unique invariant measure and for (1.1) on the whole line, the two-sided Brownian motion is invariant. On the physics level, the Brownian invariance is not hard to see, but it is a nontrivial thing to prove.

It turns out that the Edwards-Wilkinson equation $\partial_t U = \frac{1}{2}\Delta U + \xi$ admits the same invariant measure. To see this, the solution can be written as $U(t,x) = \int_0^t \int q_{t-s}(x-y)\xi(s,y) dy ds$, where q is the standard heat kernel (on $\mathbb{T}$ or $\mathbb{R}$). Since it is a Gaussian process, one can check rather easily that $U(t,x) - U(t,0)$ or $\nabla U(t,x)$ converges in law as $t \to \infty$ to the right object. Alternatively, in the periodic setting, one can also write everything in the Fourier domain: $\xi(t,x) = \sum_n B_n'(t) \frac{1}{\sqrt{L}} e^{i2\pi nx/L}$ where $\{B_n\}$ is a sequence of i.i.d. complex BMs, and each mode $\widehat{\nabla U}_n(t) = \frac{1}{\sqrt{L}} \int_0^L \nabla U(t,x) e^{-i2\pi nx/L}$ solves an Ornstein Uhlenbeck SDE which satisfies the fluctuation-dissipation relation. As a result, the invariant measure for the $\widehat{\nabla U}_n(t)$ are i.i.d. complex Gaussian, so ∇U is a white noise.

The fact that the linear and nonlinear equations admit the same invariant measure can also be seen from a formal generator calculation. On a physics level, one can write the generator associated with the stochastic Burgers equation (1.4) as

$\mathcal{L} = \mathcal{S} + \mathcal{A}$, where $\mathcal{S}$ corresponds to the linear part of the equation $\frac{1}{2}\Delta u + \nabla\xi$ and $\mathcal{A}$ corresponds to the nonlinear part $\frac{1}{2}\nabla u^2$. Take a functional $F(u)$ with $u = (u_x)_{x\in\mathbb{T}}$, we write on a formal level

$$\mathcal{S}F(u) = \int_{\mathbb{T}} \delta_{u_x} F(u) \tfrac{1}{2}\Delta u_x dx - \tfrac{1}{2}\int_{\mathbb{T}^2} \delta^2_{u_x u_y} F(u)\delta''(x-y)dxdy,$$
$$\mathcal{A}F(u) = \int_{\mathbb{T}} \delta_{u_x} F(u) \tfrac{1}{2}\nabla u_x^2 dx,$$

where δ_{u_x} is the derivative with respect to u_x. An integration by parts shows that

$$\langle \mathcal{S}F, G\rangle = \langle F, \mathcal{S}G\rangle, \qquad \langle \mathcal{A}F, G\rangle = -\langle F, \mathcal{A}G\rangle, \tag{1.7}$$

with the inner product with an underlying white noise measure $e^{-\frac{1}{2}\int_{\mathbb{T}} u_x^2 dx}du$:

$$\langle F, G\rangle = \int F(u)G(u)e^{-\frac{1}{2}\int_{\mathbb{T}} u_x^2 dx}du.$$

For the details of the formal calculation that leads to (1.7), see the discussion in Section 3 below. The invariance of the Gaussian measure can be easily seen as

$$\langle \mathcal{L}F, 1\rangle = \langle (\mathcal{S}+\mathcal{A})F, 1\rangle = \langle F, (\mathcal{S}-\mathcal{A})1\rangle = 0. \tag{1.8}$$

So $\mathcal{L}^*1 = 0$ which implies "1" is the invariant density. Because the inner product is with the Gaussian white noise measure, we conclude that the white noise is invariant under the stochastic Burgers equation.

The anti-symmetry of $\mathcal{A}$ guides most of the rigorous mathematical proofs so that various discrete approximations are constructed which preserves the symmetry and anti-symmetry. In other words, one finds discrete approximations of $\mathcal{S}, \mathcal{A}$ and $\langle\cdot,\cdot\rangle$, and prove (1.8) on the discrete level then pass to the limit to continuum.

It is difficult to rigorize the above argument on the continuum level, due to the fact that the stochastic Burgers equation is a singular SPDE and the u^2 term in the equation is purely symbolic. There is the very interesting work [18] on making sense of the generator, especially on how to construct its domain.

From an analytic perspective, it seems to be a natural question to prove the Brownian invariance directly starting from the SPDE (1.4). What is more important though, is to try to get more insight on the "cancellation" appeared in the evolution of the equation and in the formal calculation $\langle \mathcal{A}F, G\rangle = -\langle F, \mathcal{A}G\rangle$. By cancellation, we referred to the seemingly empty role played by the nonlinear term ∇u^2 in determining the invariant measure. One may hope that, with a good and new understanding of the Brownian invariance for the white noise case, one can start thinking about what happens in the colored noise case.

Here is a result which partly answers the above question:

Theorem 1.4. *Consider the equation (1.4) on $\mathbb{R}$, with the spatial covariance function $R(\cdot) = \delta(\cdot)$ or $R \in C_c^\infty$. Assume $u(0,\cdot)$ is an independent Gaussian process with covariance $R(\cdot)$. Take any test function $F, g \in C_c^\infty$, we have*

$$\mathbf{E}F(\langle u_t, g\rangle)\langle u_t, g\rangle = \langle R\star g, g\rangle \mathbf{E}F'(\langle u_t, g\rangle) + \mathbf{E}F'(\langle u_t, g\rangle)\Gamma_t(g), \qquad t \geq 0, \tag{1.9}$$

with

$$\Gamma_t(g) = \int_{\mathbb{R}^2} dxdy\, g(x)g'(y)\int_0^t \mathbb{E}^{\mathrm{poly}}_{t,x,y} R(X_s^1 - X_s^2)d(X_s^1 - X_s^2). \tag{1.10}$$

Here $\langle\cdot,\cdot\rangle$ is the natural $L^2(\mathbb{R})$ inner product, and the notations in the definition of $\Gamma_t(g)$ will be explained in Section 3 below. In addition,

$$\Gamma_t(g) = 0, \qquad \text{if } R(\cdot) = \delta(\cdot),$$

so in this case $\mathbf{E}F(\langle u_t, g\rangle)\langle u_t, g\rangle = \langle g, g\rangle \mathbf{E}F'(\langle u_t, g\rangle)$ which implies $\langle u_t, g\rangle \sim N(0, \langle g, g\rangle)$ and $u_t(\cdot)$ has the same law as the white noise.

The above result gives (yet another) different proof of the Brownian invariance of the KPZ equation with spacetime white noise, but what is more important to us is the integration by parts formula (1.9). For the colored noise case, in some loose sense it separates the Gaussian and non-Gaussian part of the random variable $\langle u_t, g\rangle$. For general $R(\cdot) \in C_c^\infty$, we do not expect the invariant measure to be Gaussian, nor do we expect it to admit any explicit description, but we do expect that $H(x) = \int_0^x \mathfrak{u}(y)dy$ is diffusive for large x, where $\mathfrak{u}$ is the limit of u_t as $t \to \infty$. This corresponds to the roughness exponent $\frac{1}{2}$ in the 1+1 KPZ universality class. To phrase it slightly differently, we hope to show that $\frac{1}{\sqrt{\varepsilon}}\mathfrak{u}(\frac{x}{\varepsilon})$ converges in law to some white noise as $\varepsilon \to 0$, and (1.9) seems to point to that direction.

The proof of (1.9) is by a not so standard Gaussian integration by parts (the second "trick"), and the proof of (1.10) is nontrivial which in our telling of the story is what keeps the white noise invariant. All these will be discussed in detail in Section 3 below.

1.3. **Time reversal and open KPZ.** In the previous section, we discussed invariant measures in the whole line and in the periodic setting. There is the interesting problem of imposing proper boundary conditions to mimic certain physical processes. For random growth models, a natural possible way is to freeze the slope at the boundary, which leads to the following so-called open KPZ equation

$$\begin{aligned} &\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \xi, \qquad t > 0, x \in (0,1), \\ &\partial_x h(t,0) = \mathsf{u}, \quad \partial_x h(t,1) = -\mathsf{v}. \end{aligned} \tag{1.11}$$

The corresponding Burgers equation takes the form

$$\begin{aligned} &\partial_t u = \tfrac{1}{2}\Delta u + \tfrac{1}{2}\nabla u^2 + \nabla\xi, \qquad t > 0, x \in (0,1), \\ &u(t,0) = \mathsf{u}, \quad u(t,1) = -\mathsf{v}. \end{aligned} \tag{1.12}$$

If we interpret u as the density of some interacting particle system, then imposing the boundary condition $u(t,0) = \mathsf{u}, u(t,1) = -\mathsf{v}$ corresponds to maintaining a fixed density at the boundaries by adding or removing particles.

One may ask what is the invariant measure for these systems and how does the boundary condition plays a role in determining the long time behavior. For the special case of $\mathsf{u}+\mathsf{v} = 0$, there is no competition between the two boundaries, and one may still expect a Brownian invariance. Namely, in this special case, the Brownian motion with a drift given by $\mathsf{u}(= -\mathsf{v})$ is invariant. The most interesting case is when $\mathsf{u} + \mathsf{v} \neq 0$ and two boundaries compete with each other. It is another miracle that, even in this case, the above equations admits explicit invariant measures. Define the space

$$C_0[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ continuous and } f(0) = 0\}$$

and the functional $Q : C_0[0,1] \to \mathbb{R}_+$

$$Q(h) = \mathbb{E}_B\left[e^{-\mathsf{v}h(1)+\mathsf{v}B(1)}\Big(\int_0^1 e^{-h(x)+B(x)}dx\Big)^{-\mathsf{u}-\mathsf{v}}\right]. \tag{1.13}$$

Here B is a standard Brownian motion with $B(0) = 0$, and $\mathbb{E}_B$ is the expectation on B. Define the measure π on $C_0[0,1]$ by

$$\frac{\pi(dh)}{\mu(dh)} = \frac{Q(h)}{Z},$$

where μ is the standard Wiener measure on $C_0[0,1]$ and Z is the normalization constant so that the r.h.s. is indeed a Radon-Nikodym derivative. We simplified

the notations here but π, Q, Z all depend on the parameters u, v. On the formal level one may write

$$\pi(dh) \propto Q(h) e^{-\frac{1}{2}\int_0^1 |\nabla h(x)|^2 dx}.$$

Here is the main result regarding the invariant measure for (1.11):

Theorem 1.5. *The measure π is invariant under the evolution of* (1.11).

One should ask what type of information the Radon-Nikodym derivative tells us. First, for the special case of $\mathsf{u} + \mathsf{v} = 0$,

$$Q(h) = \mathbb{E}_B e^{-\mathsf{v}h(1)+\mathsf{v}B(1)} = e^{-\mathsf{v}h(1)-\frac{1}{2}\mathsf{v}^2},$$

which by the Cameron-Martin theorem is the density of BM with drift $-\mathsf{v}$ on $[0,1]$ with respect to the standard BM. This is consistent with the previous discuss that if there is no competition between the two boundaries, the Brownian invariance holds. There are some other cases where $Q(h)$ can be written more explicitly. For example, if $-\mathsf{u} - \mathsf{v} \in \mathbb{Z}_+$, the expectation on B in (1.13) can be evaluated explicitly. Here is a special case:

$$Q(h) = \int_0^1 e^{-\frac{1}{2}h(x)+\frac{1}{2}(h(1)-h(x))} dx, \qquad \text{if } \mathsf{u} = \mathsf{v} = -\tfrac{1}{2}.$$

In this case, the height function has slope $-\frac{1}{2}$ at $x = 0$ and $\frac{1}{2}$ at $x = 1$, and the above invariant measure can be obtained by first sampling a uniform distributed r.v. from $[0,1]$, denoted by U, then add the drift $-\frac{1}{2}$ to the BM on $[0,U]$ and the drift $\frac{1}{2}$ on $[U,1]$. In a loose sense one may interpret U as the location of a shock, which sounds very satisfying.

To prove Theorem 1.5, a natural approach is to find appropriate discrete models with enough information on the invariant measures. This was the approach initiated in [11], which eventually leads to the discovery of the formula in (1.13) [2, 4].

The new approach we developed in proving this result is based on a time reversal of the Markov process associated with the solution to (1.12) with $\mathsf{u} = \mathsf{v} = 0$. In this special case (and also in the case of periodic boundary condition and the whole line setting), the invariant measure is Gaussian, and one can show that, at stationarity, the time reversal $\hat{u}(s,x) := u(t-s,x)$ satisfies almost the same equation

$$\partial_s \hat{u} = \tfrac{1}{2}\Delta \hat{u} - \tfrac{1}{2}\nabla \hat{u}^2 + \nabla \hat{\xi}, \tag{1.14}$$

where $\hat{\xi}$ is another spacetime white noise. Compare (1.14) with (1.12), the only difference is the sign in front of the nonlinear term. On the physics level, it is not hard to convince oneself that this is simply the consequence of the anti-symmetry of $\mathcal{A}$ in (1.7).

To show certain measures are invariant, one needs to understand the evolution of the law, and an obvious way of doing it is to consider the Fokker-Planck equation, which involves $\mathcal{L}^*$, the adjoint of the generator associated with (1.12). For a stationary Markov process, $\mathcal{L}^*$ is the generator of its time reversal, which is the third "trick" we will discuss in some details in Section 4. Roughly speaking, we will try to understand the evolution of the law of solutions to (1.12) and the time reversal trick allows us to write the law more or less explicitly by some version of the Feynman-Kac formula.

1.4. **Summary.** The examples above, together with the results of Theorems 1.3, 1.4, and 1.5, illustrate what I regard as a more analytic approach to understanding the large-scale behavior of the KPZ equation (1.1). In the remaining sections of these lecture notes, I will focus on the three main tools highlighted in the previous

discussions: the Clark–Ocone formula (Section 2), Gaussian integration by parts (Section 3), and time reversal (Section 4). I will explain in greater detail how these tools are applied to the KPZ equation, as well as to several related problems.

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2. Lecture 2: Clark-Ocone

We start with a classical result in stochastic calculus. Let $B = \{B_t\}_{t\geq 0}$ be a standard Brownian motion built on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Assuming $\{\mathcal{F}_t\}_{t\geq 0}$ is the usual filtration generated by B, then for any random variable $X \in L^2(\Omega, \mathcal{F}_t, \mathbf{P})$, there exists a unique progressive process h_s such that $\mathbf{E}\int_0^t h_s^2 ds < \infty$ and

$$X = \mathbf{E}X + \int_0^t h_s dB_s.$$

The above result is classical, and the proof can be found e.g. in [20, Theorem 5.18]. It says nothing about what h is or how to construct h from X, though. The Clark-Ocone formula expresses h in terms of a conditional expectation of the Malliavin derivative of X with respect to the white noise associated with the Brownian motion B.

To introduce the Clark-Ocone formula we need some basics of Malliavin calculus. For these lectures, all we need to know is (i) how to compute the Malliavin derivative of a certain type of smooth random variables and (ii) an integration by parts formula. For a more serious introduction to Malliavin calculus we refer to the classical monograph [22].

2.1. A crash course on Malliavin calculus.

To introduce the definition, we still consider the simple probability space generated by the Brownian motion B. Denote $\xi = B'$ which is its distributional derivative and the white noise. We use $\langle \cdot, \cdot \rangle$ to denote the natural pairing between $L^2(\mathbb{R}_+)$ and ξ, namely $\langle f, \xi \rangle = \int_0^\infty f_s dB_s$ is the Wiener integral of f against B. By smooth random variables we refer to those of the form

$$X = F(\langle f_1, \xi \rangle, \dots, \langle f_n, \xi \rangle)$$

for some $f_i \in L^2(\mathbb{R}_+)$ and some smooth F. For such random variables the Malliavin derivative is defined as the $L^2(\mathbb{R}_+)$−valued random variables

$$\mathcal{D}X = \mathcal{D}F(\langle f_1, \xi \rangle, \dots, \langle f_n, \xi \rangle) = \sum_{i=1}^n \partial_i F(\langle f_1, \xi \rangle, \dots, \langle f_n, \xi \rangle) f_i. \tag{2.1}$$

Sometimes we write $\mathcal{D}X = (\mathcal{D}_r X)_r$ with $\mathcal{D}_r X = \sum_{i=1}^n \partial_i F(\langle f_1, \xi \rangle, \dots, \langle f_n, \xi \rangle) f_i(r)$. One should think of $\mathcal{D}_r X$ as describing how the random variable X depends on the infinitesimal increments of B at r. Suppose we write $X = X(\xi)$ to indicate that X is a measurable function of ξ, then on the very formal level, $\mathcal{D}_r X = \lim_\varepsilon \frac{X(\xi + \varepsilon \delta_r) - X(\xi)}{\varepsilon}$, where δ_r is the Dirac function supported at r.

The Gaussian integration by parts formula is given by

$$\mathbf{E}[X\langle g, \xi \rangle] = \mathbf{E}\langle \mathcal{D}X, g \rangle, \qquad g \in L^2(\mathbb{R}_+) \tag{2.2}$$

where X is any smooth random variable. Here we abused the notation so on the r.h.s. $\langle \cdot, \cdot \rangle$ is the $L^2(\mathbb{R}_+)$ inner product. The above identity is nothing but a fancier

version of the well-known formulas

$$\mathbf{E}F(Z)Z = \mathbf{E}F'(Z), \qquad Z \sim N(0,1),$$
$$\mathbf{E}F(Z_1)Z_2 = \Sigma_{12}\mathbf{E}F'(Z_1), \qquad Z_1, Z_2 \sim N(0,1), \mathrm{Cov}[Z_1, Z_2] = \Sigma_{12}$$

One way to prove (2.2) is by Cameron-Martin theorem: we consider the change of measure by $e^{s\langle g,\xi\rangle - \frac{1}{2}s^2\langle g,g\rangle}$. By sending $s \to 0$ we write the l.h.s. as

$$\mathbf{E}[X\langle g,\xi\rangle] = \lim_s \tfrac{1}{s}\mathbf{E}X\big(e^{s\langle g,\xi\rangle - \frac{1}{2}s^2\langle g,g\rangle} - 1\big).$$

In computing the expectation, the role of $e^{s\langle g,\xi\rangle - \frac{1}{2}s^2\langle g,g\rangle}$ can be viewed as shifting $\xi \mapsto \xi + sg$, so with $X = F(\langle f_1,\xi\rangle, \dots, \langle f_n,\xi\rangle)$, the expectation can be rewritten as

$$\mathbf{E}Xe^{s\langle g,\xi\rangle - \frac{1}{2}s^2\langle g,g\rangle} = \mathbf{E}F(\langle f_1, \xi + sg\rangle, \dots, \langle f_n, \xi + sg\rangle).$$

Subtracting X and taking $s \to 0$ we complete the proof of the integration by parts formula.

The two equations (2.1) and (2.2) (and its generalization (2.6) below) are all we need in these lectures from Malliavin calculus. The above discussion is for the white noise space generated by B', but the same applies verbatim to any isonormal Gaussian space [22, Chapter 1]. In particular, the ξ could be the spacetime white noise driven the KPZ equation.

2.2. **The Clark-Ocone formula.** Now we can state the Clark-Ocone formula

Proposition 2.1. *Assume $X \in L^2(\Omega, \mathcal{F}_t, \mathbf{P})$ and $\mathcal{D}X$ satisfying $\mathbf{E}\langle \mathcal{D}X, \mathcal{D}X\rangle < \infty$, then*

$$X - \mathbf{E}X = \int_0^t \mathbf{E}[\mathcal{D}_r X|\mathcal{F}_r]dB_r. \tag{2.3}$$

Let us first note that $\mathcal{D}_r X$ is not necessarily $\mathcal{F}_r$–measurable (consider $X = B_t^2$ so $\mathcal{D}_r X = 2B_t 1_{[0,t]}(r)$), so the conditional expectation in (2.3) is necessary for us to interpret the integral as an Itô integral.

Before proving (2.3), let us list a few examples where this formula could be useful.

(1) There are cases where X can not be written explicitly or only admits a formal expression, for example, the local time of a Brownian motion at zero up to time t. In this case sometimes it is worth trying (2.3) to write it more explicitly. The local time is a well-defined random variable which takes the formal expression $\int_0^t \delta(B_s)ds$, where $\delta(\cdot)$ is the Dirac function at the origin. Applying (2.3), one can write it as an explicit Itô integral[1]. Here is a sketch: we have $\mathcal{D}_r X = \int_0^t \mathcal{D}_r\delta(B_s)ds = \int_0^t \delta'(B_s)1_{[0,s]}(r)ds = \int_r^t \delta'(B_s)ds$. Taking the conditional expectation

$$\mathbf{E}[\mathcal{D}_r X|\mathcal{F}_r] = \mathbf{E}[\int_r^t \delta'(B_s)ds|\mathcal{F}_r] = \int_r^t \int \delta'(y)q_{s-r}(y - B_r)dyds$$
$$= \int_r^t q'_{s-r}(B_r)ds.$$

So $X - \mathbf{E}X = \int_0^t (\int_r^t q'_{s-r}(B_r)ds)dB_r$ and the r.h.s. makes perfect sense as it is written. The reason is due to the regularization coming from the conditional expectation. One can do the same thing to other "singular"

[1] problem session

objects, such as self-intersection local time, total mass of certain GMC measures, etc.

(2) One can view (2.3) as a martingale interpolation between $\mathbf{E}X$ and X itself. Suppose $X \in L^2(\Omega, \mathcal{F}_\infty, \mathbf{P})$. Define $M_s = \mathbf{E}X + \int_0^s \mathbf{E}[\mathcal{D}_r X|\mathcal{F}_r]dB_r$, we have $M_0 = \mathbf{E}X$ and $M_\infty = X$. This type of interpolation is very useful in proving functional inequalities, and the one here is very different from the usual interpolation by the OU process in Gaussian space. There is a very cute example in proving the log-Sobolev inequality[2]: for $F \in L^2(\Omega, \mathcal{F}_\infty, \mathbf{P})$,

$$\mathbf{E}F^2 \log F^2 - \mathbf{E}F^2 \log \mathbf{E}F^2 \le 2\mathbf{E}\int_0^\infty |\mathcal{D}_r F|^2 dr. \tag{2.4}$$

Sketch: consider $X_t = M_t \log M_t$ with $M_t = \mathbf{E}F^2 + \int_0^t \mathbf{E}[\mathcal{D}_r F^2|\mathcal{F}_r]dB_r$ so that $M_0 = \mathbf{E}F^2$ and $M_\infty = F^2$. The l.h.s. of (2.4) is $\mathbf{E}X_\infty - \mathbf{E}X_0$. Applying Itô formula to X. Another relevant problem is to use (2.3) to prove the standard Gaussian concentration inequality[3].

(3) The formula (2.3) leads to the variance identity:

$$\mathrm{Var}X = \int_0^t \mathbf{E}\big[|\mathbf{E}[\mathcal{D}_r X|\mathcal{F}_r]|^2\big]dr, \tag{2.5}$$

which comes from the fact that the r.h.s. of (2.3) is an Itô integral or more specifically, the fact that the increments of a martingale is uncorrelated. Removing the correlation between increments is the central task in homogenization theory, where in most cases the problem reduces to the study of an additive functional of a Markov process. Suppose we have a Markov process at stationarity $\{\omega_s\}_s$, and one is interested in proving a CLT result for $\int_0^t f(\omega_s)ds$ where f is a certain observable defined on the state space. Here are two ways of deriving an expression of the effective variance.

(i) consider $t = N \in \mathbb{Z}_+$ and write $\int_0^N f(\omega_s)ds = \sum_{n=1}^N \int_{n-1}^n f(\omega_s)ds =: \sum_{n=1}^N \eta_n$ where $\{\eta_n\}$ is a sequence of stationary random variables. Suppose η_n decorrelates rapidly, then

$$\frac{1}{N}\mathrm{Var}\sum_{n=1}^N \eta_n \to \sum_{n\in\mathbb{Z}} \mathbf{E}\eta_0\eta_n < \infty,$$

and the limiting variance of the CLT is given by $\sum_{n\in\mathbb{Z}} \mathbf{E}\eta_0\eta_n$.

(ii) consider the Poisson equation $\mathcal{L}\phi = f$ where $\mathcal{L}$ is the generator of the process. If one can solve the Poisson equation, then

$$\int_0^t f(\omega_s)ds = \phi(\omega_t) - \phi(\omega_0) - \mathscr{M}_t,$$

where $\mathscr{M}$ is a Doob's martingale with the quadratic variation expressed through the carré du champ operator

$$\langle \mathscr{M} \rangle_t = \int_0^t [L\phi^2(\omega_s) - 2\phi(\omega_s)L\phi(\omega_s)]ds.$$

Then the limiting variance is given by $\mathbf{E}[L\phi^2 - 2\phi L\phi] = -2\mathbf{E}\phi L\phi$, which is the Dirichlet form. This is the standard homogenization approach in which one finds the corrector/harmonic coordinate ϕ to turn the additive functional into a martingale plus remainders. All the gist and difficulty lies in solving the corrector equation $\mathcal{L}\phi = f$ or some approximate version of it and derive estimates on ϕ.

[2] [5], problem session

[3] problem session.

The formula (2.5) gives another expression. One can try the three different approaches to periodic KPZ problem and all of them lead to Theorem 1.1. However, only the Clark-Ocone formula leads to Theorem 1.3. I do not know other nontrivial problems for which the formula leads to new expressions. One may try to compare the corrector approach and the Clark-Ocone approach for the toy problem where ω_s is a Brownian motion on a torus $\mathbb{T} = [0,1]$ starting from uniform measure and $f : \mathbb{T} \to \mathbb{R}$ satisfies $\int_0^1 f(x)dx = 0$[4].

It is worth mentioning the connection to *noise sensitivity*. An application of Cauchy-Schwarz to (2.5) gives the two-sided inequality

$$\int_0^t |\mathbf{E}\mathcal{D}_r X|^2 dr \le \mathrm{Var} X \le \int_0^t \mathbf{E}|\mathcal{D}_r X|^2 dr.$$

The above inequality can be seen more clearly in the Fourier domain if X is decomposed into Hermite polynomials of B (we will call those Hermite polynomial "Fourier modes" by analogy). The second one $\mathrm{Var} X \le \int_0^t \mathbf{E}|\mathcal{D}_r X|^2 dr$ is the so-called Gaussian-Poincare inequality which is used in the study of concentration of measures. On the other hand, there is a large class of problems for which the Gaussian-Poincare leads to sub-optimal bounds, simply because in the interested regime, the random variables is becoming "rougher and rougher" in the sense that they depend more and more on the contribution from high order Fourier modes. As a result, the derive $\mathbf{E}|\mathcal{D}_r X|^2$ is large. As a matter of fact, for the KPZ equation, the Gaussian-Poincare inequality gives $\mathrm{Var}\, h(t,0) \le \int_0^t \int \mathbf{E}[|\mathcal{D}_{s,y} h(t,0)|^2] dyds$ where $\mathcal{D}_{s,y}$ is now the derivative with respect to the spacetime white noise at (s,y), and it is expected (and proved in some cases) that $\int_0^t \int \mathbf{E}[|\mathcal{D}_{s,y} h(t,0)|^2] dyds \sim t$ which is much larger than $t^{2/3}$ on the l.h.s. This kind of behavior is the so-called super-concentration [6] in which the usual concentration inequality leads to sub-optimal bounds and is closely related to the noise sensitivity literature. The loss in the accuracy comes from applying Cauchy-Schwarz in (2.5), and it shows that the conditional expectation there plays a crucial role in determining the size of fluctuations.

(4) Suppose $X \in L^2(\Omega, \mathcal{F}_t, \mathbf{P})$ is a smooth random variable and F is a smooth function, applying (2.3) leads to (by the chain rule $\mathcal{D}_r[F(X)] = F'(X)\mathcal{D}_r X$)

$$F(X) - \mathbf{E}F(X) = \int_0^t \mathbf{E}[F'(X)\mathcal{D}_r X|\mathcal{F}_r] dB_r.$$

Suppose one has a good understanding of X and $\mathcal{D}_r X$, the r.h.s. is a convenient starting point to analyze the fluctuations of $F(X)$ for essentially any smooth F. This trick has been used in proving Gaussian fluctuations for KPZ equation in high dimensions with weak disorders, see [15, Section 4]. Take $X = B_t$, $\mathcal{D}_r X = 1$ for $r \le t$, so $F(B_t) - \mathbf{E}F(B_t) = \int_0^t \mathbf{E}[F'(B_t)|\mathcal{F}_r] dB_r$. One may compare it with Itô's formula.

Proof of Proposition 2.1. There are two proofs of this formula. The first one is Fourier based: write X in terms of the Hermite polynomial expansion, and compute the Malliavin derivative using the property of Hermite polynomials. This requires more notations and a few more lemmas, which can be found in [22, Proposition 1.3.14]. The approach we present here is through the integration by parts formula (2.6) below and a duality argument [23, Section 2.3].

[4] Problem session.

First, by the martingale representation theorem, there exists some h so that $X = \mathbf{E}X + \int_0^t h_s dB_s$, so it only remains to identify h. Take any ϕ that is progressive and square integrable we have by Itô isometry

$$\mathbf{E}X \int_0^t \phi_s dB_s = \mathbf{E} \int_0^t \phi_s h_s ds.$$

On the other hand, apply (2.6)

$$\mathbf{E}X \int_0^t \phi_s dB_s = \mathbf{E} \int_0^t \phi_s \mathcal{D}_s X ds = \mathbf{E} \int_0^t \phi_s \mathbf{E}[\mathcal{D}_s X|\mathcal{F}_s] ds.$$

So we have

$$\mathbf{E} \int_0^t \phi_s h_s ds = \mathbf{E} \int_0^t \phi_s \mathbf{E}[\mathcal{D}_s X|\mathcal{F}_s] ds$$

for any ϕ, which implies $h_s = \mathbf{E}[\mathcal{D}_s X|\mathcal{F}_s]$. □

Remark 2.2. In the proof we used the Hilbert space structure of $L_a^2[0,t]$ which consists all progressive processes h such that $\mathbf{E}\int_0^t h_s^2 ds < \infty$.

Lemma 2.3. *Suppose X is a smooth random variable, and h is progressive such that $\int_0^t \mathbf{E}\, h_s^2 ds < \infty$, then*

$$\mathbf{E}[X \int_0^t h_s dB_s] = \mathbf{E} \int_0^t h_s \mathcal{D}_s X ds. \tag{2.6}$$

The above formula is not obvious because $\int_0^t h_s dB_s$ is not of Gaussian distribution. As a matter of fact, since h is random, it could be of any distribution, and it is unclear how to integrate by parts. The key thing here is the independence between h_s and dB_s. By the linearity of Itô integral, one may formally write $\mathbf{E}X\int_0^t h_s dB_s = \int_0^t \mathbf{E}[X h_s dB_s]$, then by conditioning on $\mathcal{F}_s$ and using (2.2) we have $\mathbf{E}[X h_s dB_s] = \mathbf{E}[h_s \mathbf{E}[X dB_s|\mathcal{F}_s]] = \mathbf{E}[h_s \mathbf{E}[\mathcal{D}_s X ds|\mathcal{F}_s]] = \mathbf{E}[h_s \mathcal{D}_s X_s] ds$, which completes the proof. To write it rigorously, it suffices to consider simple approximation.

Proof. Assume h takes the form $h_s = \sum_i \Phi_i 1_{(s_i,t_i]}(s)$ where Φ_i is $\mathcal{F}_{s_i}$–measurable, then

$$\mathbf{E}[X \int_0^t h_s dB_s] = \sum_i \mathbf{E}X\Phi_i[B_{t_i} - B_{s_i}]$$

Conditioning on $\mathcal{F}_{s_i}$ and integrate by parts we have (using (2.2) and write $B_{t_i} - B_{s_i} = \langle \xi, 1_{[s_i,t_i]} \rangle$)

$$\mathbf{E}X\Phi_i[B_{t_i} - B_{s_i}] = \mathbf{E}\Phi_i \int_{s_i}^{t_i} \mathcal{D}_r X dr$$

which implies

$$\mathbf{E}[X \int_0^t h_s dB_s] = \sum_i \mathbf{E}\Phi_i \int_{s_i}^{t_i} \mathcal{D}_r X dr = \mathbf{E} \int_0^t h_r \mathcal{D}_r X dr.$$

The general case is proved by approximation. □

2.3. **Application to periodic KPZ.** [5]

We have the following generalization of (2.3) to the case of a spacetime white noise ξ on $\mathbb{R} \times \mathbb{T}$. Assuming $X \in L^2(\Omega, \mathcal{F}_t, \mathbf{P})$, where $\mathcal{F}_t$ is the usual filtration generated by ξ up to time t. Then

$$X - \mathbf{E}X = \int_0^t \int_{\mathbb{T}} \mathbf{E}[\mathcal{D}_{s,y} X|\mathcal{F}_s] \xi(s,y) dy ds. \tag{2.7}$$

[5] Most of this section is adopted from [16, Section 3].

The goal is to take $X = h(t,x)$ in (2.7) and prove a CLT. Whether the above expression is useful or not for our purpose obviously depends on how explicitly we can compute the conditional expectation $\mathbf{E}[\mathcal{D}_{s,y}h(t,x)|\mathcal{F}_s]$. For the KPZ equation, it turns out that $\mathcal{D}_{s,y}h(t,x)$ has the interpretation as the midpoint density of a directed polymer. This can be readily seen in the discrete setting, where the random environment is represented by a sequence of i.i.d. random variables, and $\mathcal{D}_{s,y}h(t,x)$ becomes the derivative of the free energy with respect to the random variable at the spacetime point (s,y), which is precisely the quenched probability of the polymer path passing through (s,y). In our continuous setting, we have

$$\mathcal{D}_{s,y}h(t,x) = \mathcal{D}_{s,y}\log Z(t,x) = \frac{\mathcal{D}_{s,y}Z(t,x)}{Z(t,x)},$$

where Z solves the multiplicative stochastic heat equation (SHE)

$$\partial_t Z = \tfrac{1}{2}\Delta Z + Z\xi. \tag{2.8}$$

The calculation of $\mathcal{D}_{s,y}Z(t,x)$ can be done either on the level of the mild formulation of the SHE [8, Theorem 3.2] or through the formal Feynman-Kac formula and an approximation argument[6], and the answer is

$$\mathcal{D}_{s,y}Z(t,x) = G_{t,s}(x,y)Z(s,y), \tag{2.9}$$

where $G_{t,s}(x,y)$ is the Green's function of (2.8):

$$\begin{aligned}\partial_t G_{t,s}(x,y) &= \tfrac{1}{2}\Delta_x G_{t,s}(x,y) + G_{t,s}(x,y)\xi(t,x), \quad t>s,\\ G_{s,s}(x,y) &= \delta(x-y).\end{aligned}$$

Remark 2.4. To see why (2.9) holds on a formal level, we compute the derivative on the level of SHE. Using the product rule and the formal identity $\mathcal{D}_{s,y}\xi(t,x) = \delta(t-s,x-y)$, we obtain

$$\begin{aligned}\partial_t \mathcal{D}_{s,y}Z(t,x) &= \frac{1}{2}\Delta_x \mathcal{D}_{s,y}Z(t,x) + \mathcal{D}_{s,y}(\xi(t,x)Z(t,x))\\ &= \frac{1}{2}\Delta_x \mathcal{D}_{s,y}Z(t,x) + \xi(t,x)\mathcal{D}_{s,y}Z(t,x) + \delta(t-s,x-y)Z(t,x)\\ &= \frac{1}{2}\Delta_x \mathcal{D}_{s,y}Z(t,x) + \xi(t,x)\mathcal{D}_{s,y}Z(t,x) + \delta(t-s,x-y)Z(s,y), \quad t>0\\ \mathcal{D}_{s,y}Z(0,x) &= 0.\end{aligned}$$

In other words, for fixed (s,y), with $0<s<t$, the function $f(t,x) := \mathcal{D}_{s,y}Z(t,x)$ solves the SHE with the zero initial data and the extra kick forcing term $\delta(t-s,x-y)Z(s,y)$. Hence $f(t,x) = 0$ for $0 \le t < s$ and

$$\begin{aligned}\partial_t f(t,x) &= \frac{1}{2}\Delta f(t,x) + \xi(t,x)f(t,x), \qquad t>s,\\ f(s,x) &= \delta(x-y)Z(s,y).\end{aligned}$$

Using the Green's function of SHE, f can be written as $f(t,x) = G_{t,s}(x,y)Z(s,y)$, for $t>s$, so we have (2.9). This heuristic derivation is pretty formal but it shows us essentially what happens on the level of equations [8, Theorem 3.2].

By (2.9) and (2.7), we have

$$\begin{aligned}h(t,x) - \mathbf{E}\,h(t,x) &= \int_0^t \int_{\mathbb{T}} \mathbf{E}\left[\frac{G_{t,s}(x,y)Z(s,y)}{Z(t,x)}\middle|\mathcal{F}_s\right]\xi(s,y)dyds\\ &= \int_0^t \int_{\mathbb{T}} \mathbf{E}\left[\frac{G_{t,s}(x,y)Z(s,y)}{\int_{\mathbb{T}} G_{t,s}(x,y')Z(s,y')dy'}\middle|\mathcal{F}_s\right]\xi(s,y)dyds.\end{aligned} \tag{2.10}$$

[6] Problem Session [3]

In the second step, we used the fact that G is the Green's function so $Z(t,x) = \int_{\mathbb{T}} G_{t,s}(x,y')Z(s,y')dy'$ for any $s \le t$.

It remains to understand the behavior of

$$\rho_{t,x}(s,y) := \mathcal{D}_{s,y}h(t,x) = \frac{G_{t,s}(x,y)Z(s,y)}{\int_{\mathbb{T}} G_{t,s}(x,y')Z(s,y')dy'}. \tag{2.11}$$

For $Z(s,y)$ we can write it as $Z(s,y) = e^{h(s,y)-h(s,0)}e^{h(s,0)}$. The second factor $e^{h(s,0)}$ cancels in the ratio in (2.11). For the first factor, since $h(s,y) - h(s,0)$ to converge to its invariant measure, we expect that it stabilizes for $s \gg 1$. Here is a simple lemma using the time-reversal invariance of the spacetime white noise [7]: $G_{t,s}(x,y)$ has the same law as the $G_{t-s,0}(y,x)$.

Define $\tilde{G}_{t-s,0}(y,x) = G_{t,s}(x,y)$, we rewrite

$$\begin{aligned}\rho_{t,x}(s,y) &= \frac{\tilde{G}_{t-s,0}(y,x)Z(s,y)}{\int_{\mathbb{T}} \tilde{G}_{t-s,0}(y',x)Z(s,y')dy'} \\ &= \frac{e^{\tilde{h}(t-s,y)-\tilde{h}(t-s,0)}e^{h(s,y)-h(s,0)}}{\int_{\mathbb{T}} e^{\tilde{h}(t-s,y')-\tilde{h}(t-s,0)}e^{h(s,y')-h(s,0)}dy'},\end{aligned}$$

with $\tilde{G}$ independent of $Z(s,\cdot)$, $h = \log Z$ and $\tilde{h} = \log \tilde{G}$, and by the above claim $\tilde{G}_{t-s,0}(y,x) \overset{law}{=} G_{t-s,0}(y,x)$.

Now it suffices to apply Proposition 1.2 to conclude that for $s \gg 1$ and $t - s \gg 1$

$$h(s,y) - h(s,0) \approx W_1(y), \qquad \tilde{h}(t-s,y) - \tilde{h}(t-s,0) \approx W_2(y), \tag{2.12}$$

where W_1, W_2 are independent Brownian bridges on $\mathbb{T}$. Here "$\approx$" means close in distribution (one can actually couple them so they are close in the strong sense). As a result

$$\rho_{t,x}(s,y) \approx \frac{e^{W_1(y)+W_2(y)}}{\int_{\mathbb{T}} e^{W_1(y')+W_2(y')}dy'}, \qquad t-s \gg 1, s \gg 1,$$

and

$$\begin{aligned}\operatorname{Var} h(t,x) &= \int_0^t \int_{\mathbb{T}} \mathbf{E}\left[\left|\mathbf{E}[\rho_{t,x}(s,y)|\mathcal{F}_s]\right|^2\right]dyds \\ &\approx \int_0^t \int_{\mathbb{T}} \mathbf{E}\left[\left|\mathbf{E}\left[\frac{e^{W_1(y)+W_2(y)}}{\int_{\mathbb{T}} e^{W_1(y')+W_2(y')}dy'}\middle| W_1\right]\right|^2\right]dyds = t\sigma^2\end{aligned}$$

with σ^2 given by (1.5). It is worth noting that there are three copies of independent Brownian bridges in the expression in (1.5), and that just comes from the square and the conditional expectation above. To justify the "$\approx$", one can consider a burn-in period before reaching the equilibrium in (2.12): for any $\alpha \in (0,1)$,

$$\int_0^t \int_{\mathbb{T}} \mathbf{E}\left[\left|\mathbf{E}[\rho_{t,x}(s,y)|\mathcal{F}_s]\right|^2\right]dyds = \int_{t^\alpha}^{t-t^\alpha} \int_{\mathbb{T}} \mathbf{E}\left[\left|\mathbf{E}[\rho_{t,x}(s,y)|\mathcal{F}_s]\right|^2\right]dyds + o(t).$$

The above discussion gives the effective variance. To prove the CLT, one can first massage the stochastic integral on the r.h.s. of (2.10) into a martingale then apply the martingale central limit theorem. For a sketch, see [12, Section 5.1].

For a discussion on the standard homogenization approach, we refer to [16, Section 3.1].

[7] Problem Session

3. Lecture 3: Integration by parts

In the previous section, we have already seen that it is essential to have an explicit invariant measure for $h(t,\cdot)-h(t,0)$ or equivalently $u(t,\cdot)=\nabla h(t,\cdot)$ to say anything about the large scale KPZ behaviors.

Unfortunately, for the KPZ-type equations, the spacetime white noise is the only case when the invariant measure can be written explicitly. The goal of this section is to offer a new perspective on this fact, and derive the integration by parts formula (1.9).

We phrase the first result in terms of multiplicative SHE:

Theorem 3.1. *Consider Z solving $\partial_t Z=\frac{1}{2}\Delta Z+\xi Z$ where ξ is a spacetime white noise. Assume $Z(0,x)=\exp(W(x))$ with W a two-sided Brownian motion and $W(0)=0$. Then for any $t>0$,*

$$\left\{\frac{Z(t,x)}{Z(t,0)}\right\}_{x\in\mathbb{R}}\overset{law}{=}\{\exp(W(x))\}_{x\in\mathbb{R}} \tag{3.1}$$

Recall that $u=\nabla h=\nabla\log Z$, so the above result is the same as saying that the white noise is invariant for $u(t,\cdot)$ or the two-sided Brownian motion is invariant for $h(t,\cdot)-h(t,0)$.

3.1. Formal argument.

In this section, we give a formal argument of the Brownian invariance which shows what most people (including myself actually) believe is the "right" mechanism. There are probably somewhere between 5 and 10 proofs of Theorem 3.1, but almost all the other proofs are based on making this formal argument rigorous on a discrete level.

Consider the stochastic Burgers equation

$$\partial_t u=\tfrac{1}{2}\Delta u+\tfrac{1}{2}\nabla u^2+\nabla\xi, \tag{3.2}$$

which is only formal since u is distributional valued and the u^2 above is just a symbol.

View $u(t)$ as a Markov process and for the state variable we write it as $u=(u_x)$, its generator is written as $\mathcal{L}=\mathcal{S}+\mathcal{A}$, with

$$\mathcal{S}F(u)=\int\delta_{u_x}F(u)\tfrac{1}{2}\Delta u_x dx-\tfrac{1}{2}\int\delta^2_{u_xu_y}F(u)\delta''(x-y)dxdy,$$
$$\mathcal{A}F(u)=\int\delta_{u_x}F(u)\tfrac{1}{2}\nabla u_x^2dx.$$

Here the domain of integration is either $\mathbb{T}$ or $\mathbb{R}$, namely, the one without boundaries [8]. One may think of $\delta_{u_x}F(u)=\lim_\varepsilon\frac{F(u+\varepsilon\delta_x)-F(u)}{\varepsilon}$, where, with the unfortunate notation, the δ_x on the r.h.s. is the Dirac function at x. The $-\delta''(x-y)$ appearing on the r.h.s. of $\mathcal{S}F(u)$ is the spatial covariance function of $\nabla\xi$ and the corresponding term is the Itô correction term.

In the following, we consider the inner product with the white noise measure

$$\langle F,G\rangle=\int F(u)G(u)e^{-\frac{1}{2}\int u_x^2dx}du,$$

where du is the fake Lebesgue measure in infinite dimensions. The goal is to show

$$\langle\mathcal{S}F,G\rangle=\langle F,\mathcal{S}G\rangle,\qquad\langle\mathcal{A}F,G\rangle=\langle F,-\mathcal{A}G\rangle, \tag{3.3}$$

[8] Incidentally, one may try to do the same for the open KPZ equation and show (1.13) is the invariant density. This seems nontrivial though, even on the physics level.

which implies $\langle \mathcal{L}F, 1\rangle = 0$, so "1" is the invariant density which means the white noise measure is invariant under the evolution of (3.2).

To draw an analogy, here is a review of some basics in Markov chain[9]. Suppose the state space E of the Markov chain is countable and π is a probability measure on E, and P is the transition matrix. Define the inner product on $\ell^2(\pi)$ by $\langle F, G\rangle = \sum_x F(x)G(x)\pi(x)$, then $\langle LF, 1\rangle = \langle (P-I)F, 1\rangle = 0$ for any F implies that π is invariant. This is because $\langle PF, 1\rangle = \sum_x PF(x)\pi(x) = \sum_{x,y} P(x,y)F(y)\pi(x) = \sum_y (\sum_x \pi(x)P(x,y))F(y)$, and $\langle F, 1\rangle = \sum_y \pi(y)F(y)$. Since F is arbitrary, we have $\sum_x \pi(x)P(x,y) = \pi(y)$ which shows that π is invariant.

Proof of (3.3). We will only show the anti-symmetry of $\mathcal{A}$ in (3.3). The symmetry of $\mathcal{S}$ will be done in the Problem Session[10]. We write

$$\langle \mathcal{A}F, G\rangle = \int \left(\int \delta_{u_x} F(u) \tfrac{1}{2} \nabla u_x^2 dx \right) G(u) e^{-\frac{1}{2}\int u_y^2 dy} du.$$

Integrate by parts and δ_{u_x} acts on ∇u_x^2, $G(u)$ and $e^{-\frac{1}{2}\int u_x^2 dx}$ separately and we have $\langle \mathcal{A}F, G\rangle = I_1 + I_2 + I_3$ with

$$\begin{aligned} I_1 &= -\int \left(\int F(u) \tfrac{1}{2} \nabla \delta_{u_x} u_x^2 dx \right) G(u) e^{-\frac{1}{2}\int u_y^2 dy} du \\ &= -\int \left(\int F(u) \tfrac{1}{2} \nabla 2u_x \delta_0 dx \right) G(u) e^{-\frac{1}{2}\int u_y^2 dy} du = 0 \end{aligned}$$

The last "=" is because $\int \nabla u_x dx = 0$. When δ_{u_x} acts on $G(u)$,

$$I_2 = -\int \left(\int F(u) \tfrac{1}{2} \nabla u_x^2 \delta_{u_x} G(u) dx \right) e^{-\frac{1}{2}\int u_y^2 dy} du = \langle F, -\mathcal{A}G\rangle$$

The key is when δ_{u_x} acts on the Gaussian density,

$$\begin{aligned} I_3 &= -\int \left(\int F(u) \tfrac{1}{2} \nabla u_x^2 G(u) \delta_{u_x} e^{-\frac{1}{2}\int u_y^2 dy} dx \right) du \\ &= \int \left(\int F(u) \tfrac{1}{2} \nabla u_x^2 G(u) u_x e^{-\frac{1}{2}\int u_y^2 dy} dx \right) du. \end{aligned}$$

Now it suffices to realize that $(\nabla u_x^2) u_x = \frac{2}{3} \nabla u_x^3$ is a total derivative so the integration in x is zero and $I_3 = 0$. □

To really understand the above argument, it probably helps to work on the relevant discrete models and justify every corresponding step in the above calculation. See [17, Section 1] and the references cited there.

3.2. **Integration by parts on Burgers.** From now on we write $u_t = u(t, \cdot)$ and view it as a Markov process taking values in some state space, and the goal is to show that, if started with a white noise at $t = 0$, u_t is a white noise for any $t > 0$, or equivalently, for any smooth test function $\langle u_t, g\rangle = \int u_t(x) g(x) dx \sim N(0, \|g\|_2^2)$. Take another test function F, Stein's equation says it is enough to show

$$\mathbf{E} F(\langle u_t, g\rangle) \langle u_t, g\rangle = \|g\|_2^2 \mathbf{E} F'(\langle u_t, g\rangle). \tag{3.4}$$

Linear case. We first consider the trivial case when u solves the corresponding linear equation $\partial_t u = \frac{1}{2}\Delta u + \nabla \xi$, as this can be viewed as a degenerate case of the nonlinear problem where some underlying diffusion is replaced by a Brownian motion, and on this level, one really sees clearly how the argument works. There are easier ways of doing the calculation, and we choose the following one just to draw the analogy with the nonlinear setting.

[9] Problem Session.
[10] Problem session.

The initial data for u is W' which is a spatial white noise, since W is a two-sided Brownian motion. The solution can be written explicitly as (with $q_t(x) = (2\pi t)^{-1/2}\exp(-x^2/2t)$)

$$u(t,x) = q_t \star W'(x) + \int_0^t \int_{\mathbb{R}} q_{t-s}(x-y)\xi'(s,y)dyds. \tag{3.5}$$

Here we used $\xi' = \nabla\xi$. There are two independent Gaussian randomnesses on the r.h.s, and we will use the formula (2.2).

To use the integration by parts, we rewrite (3.5) as

$$\langle u_t, g\rangle_{L^2(\mathbb{R})} = \langle W', g_1\rangle_{L^2(\mathbb{R})} + \langle \xi, g_2\rangle_{L^2([0,t]\times\mathbb{R})},$$

with

$$g_1(x) = q_t \star g(x), \qquad g_2(s,y) = -\int_{\mathbb{R}} q'_{t-s}(x-y)g(y)dy,$$

where we have abused notations and the pairing $\langle\cdot,\cdot\rangle$ is to be understood accordingly. The l.h.s. of (3.4) can be written as

$$\mathbf{E}F(\langle u_t, g\rangle)\langle u_t, g\rangle = \mathbf{E}F(\langle W', g_1\rangle + \langle \xi, g_2\rangle)(\langle W', g_1\rangle + \langle \xi, g_2\rangle).$$

Integrate by parts against W' and ξ separately, we have

$$\mathbf{E}F(\langle W', g_1\rangle + \langle \xi, g_2\rangle)\langle W', g_1\rangle = \int g_1^2(x)dx\, \mathbf{E}F'(\langle W', g_1\rangle + \langle \xi, g_2\rangle),$$

$$\mathbf{E}F(\langle W', g_1\rangle + \langle \xi, g_2\rangle)\langle \xi, g_2\rangle = \int_0^t \int_{\mathbb{R}} |g_2(s,y)|^2 dyds\, \mathbf{E}F'(\langle W', g_1\rangle + \langle \xi, g_2\rangle).$$

Write in Fourier domain we have

$$\int g_1^2(x)dx = \int |\hat{g}(\xi)|^2 e^{-t|\xi|^2}d\xi,$$

$$\int_0^t \int |g_2(s,y)|^2 dyds = \int_0^t \int |\hat{g}(\xi)|^2|\xi|^2 e^{-|\xi|^2(t-s)}d\xi ds = \int |\hat{g}(\xi)|^2(1-e^{-|\xi|^2 t})d\xi.$$

The sum is precisely $\int g^2$, which implies (3.4).

Nonlinear case. In the nonlinear case we do not have anything as clean as (3.5) which gives the Gaussian factors needed for the integration by parts. One may try to write the equation as

$$\partial_t u = \tfrac{1}{2}\Delta u + u\nabla u + \nabla\xi$$

and pretend we have already solved the equation so the Green's function for $\frac{1}{2}\Delta + u\nabla$ is known which we denote as $\mathcal{G}_{t,s}(x,y)$. Using $\mathcal{G}$ the solution can be written as

$$u(t,x) = \int_{\mathbb{R}} \mathcal{G}_{t,0}(x,y)W'(y)dy + \int_0^t \int_{\mathbb{R}} \mathcal{G}_{t,s}(x,y)\xi'(s,y)dyds, \tag{3.6}$$

which is to be compared with (3.5). There are a bunch of problems with the above integrals: a priori it is unclear how they should be defined since $\mathcal{G}$ depends on ξ and W'. Even if we make sense of the integrals, it is unclear how to do Gaussian integration by parts using either (2.2) or (2.6).

The approach we adopt here is through the Hopf-Cole and the Feynman-Kac formula: since $u = (\log Z)' = Z^{-1}Z'$ and on the formal level Z can be written as

$$Z(t,x) = \mathbb{E}_B\left[e^{\int_0^t \xi(t-s,x+B_s)ds} e^{W(x+B_t)}\right], \tag{3.7}$$

where B is a standard BM starting at the origin independent of everything else. Note that the random variable $\int_0^t \xi(t-s,x+B_s)ds$ is not well-defined when ξ is a spacetime white noise, but for the moment we will assume ξ is smooth in the spatial

variable, and for each fixed realization of the Brownian motion, $\int_0^t \xi(t-s, x+B_s)ds$ is a well-defined Wiener integral[11].

With Z given by (3.7), one have the following probabilistic representation of solutions to the stochastic Burgers equation

$$u(t,x) = \frac{Z'(t,x)}{Z(t,x)} = \frac{\mathbb{E}_B\left[e^{\int_0^t \xi(t-s,x+B_s)ds} e^{W(x+B_t)}\left[\int_0^t \xi'(t-s,x+B_s)ds + W'(x+B_t)\right]\right]}{Z(t,x)}. \tag{3.8}$$

Again, it is a formal expression for singular ξ or W', but one can proceed by approximation and deal with smooth noise and smooth initial data first.

3.2.1. *A Crash course on directed polymer in random environment.* [12] In this section, we introduce the model of directed polymers in a random environment, which is a prototypical model in the study of disordered systems, and is closely related to the KPZ equation. A classical reference is the monograph [9].

In the discrete setting, consider the lattice points $\mathbb{Z}_+ \times \mathbb{Z}^d$, with a sequence of i.i.d. random variables $\omega = \{\omega_{i,j}\}_{i\in\mathbb{Z}_+, j\in\mathbb{Z}^d}$. Let $\{S_n\}_n$ be a SSRW starting at the origin. For each n and each realization of ω, define the energy

$$H_n(\omega, S) = \sum_{i=1}^n \omega_{i,S_i}$$

and the partition function $Z_\beta(n,\omega) = \sum_S e^{\beta H_n(\omega,S)}$, where the summation is over the 2^n random walk paths. Define the Gibbs measure on the paths of length n as the SSRW measure tilted by $\frac{e^{\beta H_n(\omega,S)}}{Z_\beta(n,\omega)}$, where β is the inverse temperature. This is the quenched polymer measure, and when we say "directed polymer", we mean the path under such a measure.

In the continuous setting, one can replace the i.i.d. environment by a Gaussian process ξ and the SSRW by a Brownian motion B. The corresponding energy takes the form

$$H_t(B,\xi) = \int_0^t \xi(s, B_s)ds$$

and the partition function is $Z_\beta(t,\xi) = \mathbb{E}_B\left[e^{\beta H_t(B,\xi)}\right]$. The polymer measure is simply the Wiener measure tilted by $\frac{e^{\beta H_t(B,\xi)}}{Z_\beta(t,\xi)}$. The connection to the solution of multiplicative SHE is through a time reversal of the random environment and the Feynman-Kac formula. If we define $Z(t,x)$ to be the partition function of the directed polymer in the environment $\xi(t-\cdot,\cdot)$, with the path starting at x, then $Z(t,x) = \mathbb{E}_B e^{\beta\int_0^t \xi(t-s,x+B_s)ds}$, and it solves

$$\partial_t Z = \tfrac{1}{2}\Delta Z + \beta Z\xi \tag{3.9}$$

with $Z(0,x) \equiv 1$. With different initial data, we view the corresponding Z as the partition function of the polymer measure with the corresponding boundary condition.

It is instructive to view the polymer process as a diffusion in a random environment. Assuming everything is smooth for the sake of discussion. We need to introduce some notations. Let Z be the solution to (3.9) with non-negative initial

[11] Problem Session, Feynman-Kac representation for the solution to SHE [3].
[12] Problem Session.

data Z_0, and define $\mathbf{P}^{\text{poly}}_{t,x}$ to be the quenched polymer measure on $C[0,t]$, with the (fixed) environment $\xi(t-\cdot,\cdot)$ and starting point x: for any subset $\Lambda \subset C[0,t]$,

$$\mathbf{P}^{\text{poly}}_{t,x}(\Lambda) = \frac{\mathbb{E}_B\big[1_\Lambda(x+B_\cdot)e^{\beta\int_0^t \xi(t-s,x+B_s)ds} Z_0(x+B_t)\big]}{Z(t,x)}. \tag{3.10}$$

At the same time, define $u = \nabla \log Z$, and consider the diffusion process X_s given by the SDE

$$dX_s = u(t-s,X_s)ds + dB_s, \qquad X_0 = x. \tag{3.11}$$

Let $\mathbf{Q}^{\text{diff}}_{t,x}$ be the quenched law of $(X_s)_{s\in[0,t]}$. By quenched, we mean for each fixed realization of ξ (hence each fixed realization of u).

The following result states that one can view the directed polymer as a diffusion in the Burgers environment.

Lemma 3.2. $\mathbf{P}^{\text{poly}}_{t,x} = \mathbf{Q}^{\text{diff}}_{t,x}$.

The lemma will be proved through a Girsanov transformation[13]. To get a feeling why it should be true from a PDE perspective, one may consider the marginal of the measure at time t. For any $A \subset \mathbb{R}$, consider the measure $\mu_{t,x}$ on $\mathbb{R}$ defined as $\mu_{t,x}(A) = \mathbf{P}^{\text{poly}}_{t,x}(\{\omega : \omega_t \in A\})$ so

$$\mu_{t,x}(A) = \frac{\mathbb{E}_B\big[1_A(x+B_t)e^{\beta\int_0^t \xi(t-s,x+B_s)ds} Z_0(x+B_t)\big]}{Z(t,x)} =: \frac{\tilde{Z}(t,x)}{Z(t,x)}$$

It is clear that $\tilde{Z}$ and Z solve the same equation (3.9), but with different initial data $\tilde{Z}(0,x) = 1_A(x)Z_0(x)$ and $Z(0,x) = Z_0(x)$. It is an exercise of chain-rule to check that $\mu_{t,x}(A)$ solves the PDE

$$\partial_t \mu_{t,x}(A) = \tfrac{1}{2}\Delta\mu_{t,x}(A) + \nabla \log Z(t,x)\nabla\mu_{t,x}(A), \qquad \mu_{0,x}(A) = 1_A.$$

By the probabilistic representation of the above PDE, we have

$$\mu_{t,x}(A) = \mathbb{E}_B 1_A(X_t),$$

with X given by (3.11). This shows that $\mu_{t,x}(A) = \mathbf{Q}^{\text{diff}}_{t,x}(\{\omega : \omega_t \in A\})$.

The Feynman-Kac formula and the diffusion process (3.11) provide two probabilistic representations. There is another one given in [10], which seems to be genuinely different, and it would be interesting to investigate whether it could be of any use in this context.

By Lemma 3.2 and (3.8), we can write (assuming everything is smooth)

$$u(t,x) = \mathbb{E}_B\bigg[\int_0^t \xi'(t-s,X_s)ds + W'(X_t)\bigg],$$

which may be viewed as a probabilistic counterpart of (3.6).

3.2.2. *Integration by parts on smooth Burgers.* With the above preparation, in particular, the formula (3.8), we can start integrating by parts on the stochastic Burgers equation. From now on, we assume that ξ is white in time and colored in space with covariance function $R \in C_c^\infty$. In particular we assume it is constructed from a spacetime white noise η through a spatial convolution:

$$\xi(t,x) = \int_{\mathbb{R}} \phi(x-y)\eta(t,y)dy,$$

[13] Problem Session

with a symmetric $\phi \in \mathcal{C}_c^\infty$ so that $\phi \star \phi = R$. We assume W' is smoothed in the same way:

$$W'(x) = \int_{\mathbb{R}} \phi(x-y)\zeta(y)dy,$$

where ζ is a spatial white noise.

We start from the l.h.s. of (3.4) $\mathbf{E}F(\langle u_t, g\rangle)\langle u_t, g\rangle$. There are two places u_t appearing in it, but we will write the first one as $\langle u_t, g\rangle = -\langle h_t, g'\rangle$ where h_t is the solution to KPZ so that

$$\mathbf{E}F(\langle u_t, g\rangle)\langle u_t, g\rangle = \mathbf{E}F(-\langle h_t, g'\rangle)\langle u_t, g\rangle.$$

The reason is purely technical since the calculation of Malliavin derivative is easier for h than u. Since the underlying randomness is the spacetime white noise η and the spatial white noise ζ, we will express everything in terms of them.

First, to simplify the notation define

$$\begin{aligned} H_{t,x}(B,\xi,W) &= \int_0^t \xi(t-s, x+B_s)ds + W(x+B_t) \\ &= \langle \phi(x+B_{t-\cdot}-\cdot), \eta\rangle_{L^2([0,t]\times\mathbb{R})} + \langle \phi \star 1_{[0,x+B_t]}(\cdot), \zeta\rangle_{L^2(\mathbb{R})}, \end{aligned}$$

hence

$$\partial_x H_{t,x}(B,\xi,W) = \langle \phi'(x+B_{t-\cdot}-\cdot), \eta\rangle_{L^2([0,t]\times\mathbb{R})} + \langle \phi(x+B_t-\cdot), \zeta\rangle_{L^2(\mathbb{R})}.$$

The solution is given by

$$\begin{aligned} u_t(x) =& Z_t(x)^{-1}\mathbb{E}_B\big[e^{H_{t,x}(B,\xi,W)}\partial_x H_{t,x}(B,\xi,W)\big] \\ =& Z_t(x)^{-1}\mathbb{E}_B\big[e^{H_{t,x}(B,\xi,W)}\langle \phi'(x+B_{t-\cdot}-\cdot), \eta\rangle_{L^2([0,t]\times\mathbb{R})}\big] \\ &+ Z_t(x)^{-1}\mathbb{E}_B\big[e^{H_{t,x}(B,\xi,W)}\langle \phi(x+B_t-\cdot), \zeta\rangle_{L^2(\mathbb{R})}\big] =: u_{1,t}(x) + u_{2,t}(x). \end{aligned} \tag{3.12}$$

The analysis of $u_{1,t}$ is apparently more complicated since it involves the integration in time (in the inner product).

To state the result, we introduce another notation: $\mathbb{E}^{\mathrm{poly}}_{t,x}$ is the expectation on the measure $\mathbf{P}^{\mathrm{poly}}_{t,x}$ defined in (3.10), and $\mathbb{E}^{\mathrm{poly}}_{t,x_1,x_2}$ is the expectation on the product measure $\mathbf{P}^{\mathrm{poly}}_{t,x_1} \times \mathbf{P}^{\mathrm{poly}}_{t,x_2}$. So for any bounded $F : C[0,t] \times C[0,t] \to \mathbb{R}$, we have

$$\begin{aligned} &\mathbb{E}^{\mathrm{poly}}_{t,x_1,x_2}F(X_1,X_2) \\ &= \frac{\mathbb{E}_{B_1,B_2}F(x+B_1, x+B_2)\prod_{j=1}^2 e^{\int_0^t \xi(t-s,x+B_j(s))ds + W(x+B_j(t))}}{Z_t(x_1)Z_t(x_2)}. \end{aligned}$$

The following is obtained by using (2.2) for η:

$$\mathbf{E}F(-\langle h_t, g'\rangle)u_{1,t}(x) = -\mathbf{E}F'(\langle u_t, g\rangle)\int_{\mathbb{R}} dw\, g'(w)\mathbb{E}^{\mathrm{poly}}_{t,x,w}\int_0^t R'(X_s^1 - X_s^2)ds. \tag{3.13}$$

Instead of deriving (3.13), we will prove the following result, which is a special case of (3.13) (with $F(x) = x$ and g' chosen as a Dirac function). It is straightforward to follow the same proof verbatim to derive (3.13).

Lemma 3.3. *With u_1 defined in (3.12), for any $x_1, x_2 \in \mathbb{R}$ and $t > 0$,*

$$\mathbf{E}\big[u_{1,t}(x_1)h_t(x_2)\big] = \mathbf{E}\mathbb{E}^{\mathrm{poly}}_{t,x_1,x_2}\int_0^t R'(X_s^1 - X_s^2)ds. \tag{3.14}$$

The proof of the above lemma is the most technical one presented in these notes, but one should keep in mind that there are no estimates and only identities. To see why R' appears in the result, we note that h is forced by ξ while u is forced by ξ', so the covariance function of ξ' and ξ is $\mathbf{E}[\xi'(t,x)\xi(s,y)] = \delta(t-s)R'(x-y)$.

Proof of Lemma 3.3. We write

$$\mathbf{E}u_{1,t}(x_1)h_t(x_2) = \mathbb{E}_{B_1}\mathbf{E}Z_t(x_1)^{-1}e^{H_{t,x_1}(B^1,\xi,W)}h_t(x_2)\langle\phi'(x_1+B^1_{t-\cdot}-\cdot),\eta\rangle_{L^2([0,t]\times\mathbb{R})}$$

and compute the expectation $\mathbf{E}$ first. Here B^1 is a standard BM and we perform the inner expectation $\mathbf{E}$ for each realization of B^1.

Define $X = X_1X_2X_3 = Z_t(x_1)^{-1}e^{H_{t,x_1}(B^1,\xi,W)}h_t(x_2)$, so by (2.2)

$$\mathbf{E}X\langle\phi'(x+B^1_{t-\cdot}-\cdot),\eta\rangle_{L^2([0,t]\times\mathbb{R})} = \mathbf{E}\langle\mathcal{D}X,\phi'(x+B^1_{t-\cdot}-\cdot)\rangle_{L^2([0,t]\times\mathbb{R})}.$$

To compute $\mathcal{D}X$ we use the product rule

$$\mathcal{D}_{s,y}X = \sum_i(\mathcal{D}_{s,y}X_i)\prod_{j\neq i}X_j.$$

There are three terms. For $\mathcal{D}X_3 = \mathcal{D}h_t(x_2)$, we have

$$\begin{aligned}\mathcal{D}_{s,y}h_t(x_2) &= \mathcal{D}_{s,y}\log Z_t(x_2) = Z_t(x_2)^{-1}\mathcal{D}_{s,y}Z_t(x_2)\\ &= Z_t(x_2)^{-1}\mathbb{E}_{B^2}e^{H_{t,x_2}(B^2,\xi,W)}\phi(x_2+B^2_{t-s}-y).\end{aligned}$$

Here in the last step when computing $\mathcal{D}Z$ we used the expression

$$Z_t(x) = \mathbb{E}_{B^2}\big[e^{H_{t,x}(B^2,\xi,W)}\big]$$

and interchanged the order of $\mathcal{D}$ and $\mathbb{E}_{B^2}$. The use of B^2 is due to the fact that B^1 is already used. As a result,

$$\begin{aligned}&\mathbf{E}\big[X_1X_2\langle\mathcal{D}X_3,\phi'(x_1+B^1_{t-\cdot}-\cdot)\rangle_{L^2([0,t]\times\mathbb{R})}\big]\\ &= \mathbf{E}\mathbb{E}_{B^2}\prod_{j=1}^2 Z_t(x_j)^{-1}e^{H_{t,x_j}(B^j,\xi,W)}\langle\phi(x_2+B^2_{t-\cdot}-\cdot),\phi'(x_1+B^1_{t-\cdot}-\cdot)\rangle_{L^2([0,t]\times\mathbb{R})}\\ &= \mathbf{E}\mathbb{E}_{B^2}\prod_{j=1}^2 Z_t(x_j)^{-1}e^{H_{t,x_j}(B^j,\xi,W)}\int_0^t R'(x_1+B^1_s-x_2-B^2_s)ds.\end{aligned}$$

Further taking expectation in B^1, the r.h.s. is precisely $\mathbf{E}\mathbb{E}^{\mathrm{poly}}_{t,x_1,x_2}\int_0^t R'(X^1_s-X^2_s)ds$.

We claim that the other two terms are identically zero[14]. As a matter of fact, it suffices to note that, by exactly the same calculation, what would appear is either $\mathbb{E}^{\mathrm{poly}}_{t,x_1,x_1}\int_0^t R'(X^1_s-X^2_s)ds$ or $\mathbb{E}^{\mathrm{poly}}_{t,x_1,x_1}\int_0^t R'(0)ds$. Since R is even, R' is odd, so $R'(0)=0$ and $\mathbb{E}^{\mathrm{poly}}_{t,x_1,x_1}\int_0^t R'(X^1_s-X^2_s)ds = 0$ by symmetry. This ends the proof. □

The identity (3.14) is surprising. One should compare it with what happens in the linear case. Suppose h solves $\partial_t h = \frac{1}{2}\Delta h+\xi$ and $u = h'$, with $h_0\equiv 0$, then

$$\mathbf{E}[u_t(x_1)h_t(x_2)] = \mathbb{E}_{x_1,x_2}\int_0^t R'(X^1_s-X^2_s)ds, \tag{3.15}$$

where $\mathbb{E}_{x_1,x_2}$ are the expectation on two independent BMs X^1,X^2 starting at x_1,x_2 respectively. Compared to the expression in (3.12), in this case we have the probabilistic representation

$$\begin{aligned}u_t(x_1) &= \mathbb{E}_B\langle\phi'(x_1+B_{t-\cdot}-\cdot),\eta\rangle_{L^2([0,t]\times\mathbb{R})},\\ h_t(x_2) &= \mathbb{E}_B\langle\phi(x_2+B_{t-\cdot}-\cdot),\eta\rangle_{L^2([0,t]\times\mathbb{R})},\end{aligned}$$

in which case the polymer measure is replaced by the Wiener measure, and it is enough to apply Itô isometry to derive (3.15). Compare to (3.14) we observe that the only difference is the Brownian motions being replaced by directed polymers.

In a loose sense, all the difficulties in KPZ compared to Edwards-Wilkinson comes from the fact that the Brownian motion which we understand well is replaced by the directed polymer in a random environment.

[14] Problem Session.

One may integrate in parts on $u_{2,t}$ using the Gaussianity of ζ to obtain an expression for $\mathbf{E}F(-\langle h_t, g'\rangle)u_{2,t}(x)$. We will not do it in these notes, and the interested readers may find it in [17, Section 5].

3.2.3. *Removing mollification and Itô formula.* To see how to piece everything together as we remove the smoothing, namely replace $R = R_\varepsilon$ with $R_\varepsilon(\cdot) \to \delta(\cdot)$, there is a nontrivial application of Itô's formula. On the formal level, in (3.14), we will be dealing with $\int_0^t \delta'(X_s^1 - X_s^2)ds$, which looks scary since it does not make any sense. However, one should keep in mind there is an expectation $\mathbb{E}^{\text{poly}}_{t,x_1,x_2}$ outside.

Take the linear case (3.15) to warm up, if we apply Itô's formula (when X^1, X^2 are independent Brownian motions)

(3.16)
$$\int_0^t \delta'(X_s^1 - X_s^2)ds = \tfrac{1}{2}\text{sgn}(X_t^1 - X_t^2) - \tfrac{1}{2}\text{sgn}(X_0^1 - X_0^2) - \int_0^t \delta(X_s^1 - X_s^2)d(X_s^1 - X_s^2),$$

while the Ito integral term does not make sense, we take the expectation to kill it. In other words, in this case,

$$\mathbf{E}[u_t(x_1)h_t(x_2)] = \mathbb{E}_{x_1,x_2}\tfrac{1}{2}\text{sgn}(X_t^1 - X_t^2) - \tfrac{1}{2}\text{sgn}(x_1 - x_2).$$

For fixed x_1, x_2, the first term goes to zero as $t \to \infty$[15], which implies

$$\mathbf{E}u_t(x_1)h_t(x_2) \to -\tfrac{1}{2}\text{sgn}(x_1 - x_2)$$

and taking derivative in x_2 on both sides leads to

$$\mathbf{E}u_t(x_1)u_t(x_2) \to \delta(x_1 - x_2)$$

which is the covariance function of a spatial white noise.

For the nonlinear case, (3.16) still holds, because under the polymer measure, X^1, X^2 are both semi-martingales with $\langle X^1\rangle_t = \langle X^2\rangle_t = t$, which can be seen from the diffusion representation (3.11). However, it is less obvious what happens to

$$\mathbb{E}^{\text{poly}}_{t,x_1,x_2}\int_0^t \delta(X_s^1 - X_s^2)d(X_s^1 - X_s^2).$$

Suppose we take the point of view of diffusion in random environment so

$$dX_s^i = u(t-s, X_s^i)ds + dB_s^i, \qquad i = 1, 2,$$

then the above quantity can be rewritten as

$$\mathbb{E}^{\text{diff}}_{t,x_1,x_2}\int_0^t \delta(X_s^1 - X_s^2)[u(t-s, X_s^1) - u(t-s, X_s^2)]ds.$$

The dB_s^i terms do not show up because they average to zero under $\mathbb{E}^{\text{diff}}_{t,x_1,x_2}$. If u were function-valued, it is not hard to convince oneself that the above terms equal to zero because the only contribution to $\delta(X_s^1 - X_s^2)$ is when $X^1 = X^2$ in which case the difference between the drift is zero. However, in the spacetime white noise case, u is not function-valued, so it is not obvious how the argument should work. This turns out to be the most technical part of the paper.

What we did in the paper is a path exchange argument. To show that

$$\mathbb{E}^{\text{poly}}_{t,x_1,x_2}\int_0^t R_\varepsilon(X_s^1 - X_s^2)d(X_s^1 - X_s^2) \to 0, \quad \text{as } \varepsilon \to 0, \tag{3.17}$$

the key is to use the symmetry in the above expression: if $x_1 = x_2$, then X^1, X^2 are exchangeable, and since the whole expression is odd in $X^1 - X^2$, it must be zero. For $x_1 \neq x_2$, we wait until the two paths to meet, and right after the meet, they become exchangeable which does not contribute to the above integral. Before they

[15] In the discussion of the linear case, we didn't start from stationarity, otherwise this term will be cancelled by the term coming from the integration by parts of the initial data.

meet, to contribute to the integral they have to be within ε−distance from each other since R_ε is supported on an ε−ball centered at the origin. But as $\varepsilon \to 0$, the "intersection" time is vanishing. Combine the small amount of intersection time before meeting and the exchangeability after meeting, we conclude (3.17).

The proof guided by the above discussion can be found in [17, Section 4].

The estimate (3.17) provides another way of interpreting the Brownian invariance of the KPZ equation, and explains why the nonlinear problem has the same invariant measure as the linear problem from a more probabilistic perspective, which is different from generator point of view in Section 3.1. For the linear problem, the l.h.s. is exactly zero since X^1, X^2 are BMs (martingales); for the nonlinear problem, the processes are directed polymers and there is a random drift contributing on the l.h.s, but as the noise becomes white the correlation length is vanishing, which makes the contribution vanish. The correlation in KPZ arises from polymer paths intersection/coalescence, which is a hard problem to study by itself. This new argument presented here suggests the quantities $\mathbb{E}^{\text{poly}}_{t,x_1,x_2} \int_0^t R'(X_s^1 - X_s^2)ds$ or $\mathbb{E}^{\text{poly}}_{t,x_1,x_2} \int_0^t R(X_s^1 - X_s^2)d(X_s^1 - X_s^2)$ as natural objects one may use to study the correlation.

4. Lecture 4: Time reversal

In this section, we discuss the study on the invariant measure for the open KPZ equation:

$$\begin{aligned} &\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \xi, \qquad t>0, x\in(0,1),\\ &\partial_x h(t,0) = \mathsf{u}, \quad \partial_x h(t,1) = -\mathsf{v}, \end{aligned} \tag{4.1}$$

where $\mathsf{u},\mathsf{v}\in\mathbb{R}$ are fixed constants and ξ is a spacetime white noise. The goal is to show that the invariant measure is given by the measure π on $C_0[0,1]$ which is absolutely continuous respect to the Wiener measure with the Radon-Nikodym derivative

$$Q(h) = \mathbb{E}_B\Big[e^{-\mathsf{v}h(1)+\mathsf{v}B(1)}\big(\int_0^1 e^{-h(x)+B(x)}dx\big)^{-\mathsf{u}-\mathsf{v}}\Big]. \tag{4.2}$$

Here B is a standard Brownian motion with $B(0)=0$, and $\mathbb{E}_B$ is the expectation on B.

4.1. **Cameron-Martin.** The following trick turns the boundary conditions into homogeneous Neumann boundary conditions. Assuming everything is smooth and consider $Z=e^h$ which solves[16]

$$\begin{aligned} &\partial_t Z = \tfrac{1}{2}\Delta Z + Z\xi,\\ &\partial_x Z(t,0) = \mathsf{u}Z(t,0), \quad \partial_x Z(t,1) = -\mathsf{v}Z(t,1). \end{aligned} \tag{4.3}$$

Take any smooth test function $g:[0,1]\to\mathbb{R}$ with $g'(0)=g'(1)=0$, we have

$$\begin{aligned} \langle \Delta Z_t, g\rangle &= \int_0^1 \Delta Z_t(x)g(x)dx = -\int_0^1 \partial_x Z_t(x)g'(x)dx + \partial_x Z_t(x)g(x)|_0^1\\ &= \int_0^1 Z_t(x)g''(x)dx - \mathsf{v}Z_t(1)g(1) - \mathsf{u}Z_t(0)g(0), \end{aligned}$$

where we used the Robin boundary condition in the last step. The boundary terms can be written as

$$-\mathsf{v}Z_t(1)g(1) - \mathsf{u}Z_t(0)g(0) = \langle Z_t(-\mathsf{v}\delta_1 - \mathsf{u}\delta_0), g\rangle,$$

where δ_0,δ_1 are Dirac functions at $0,1$ respectively (really at $0+$ and $1-$). Therefore, (4.3) is equivalent with

$$\begin{aligned} &\partial_t Z = \tfrac{1}{2}\Delta Z + Z\xi + \tfrac{1}{2}Z(-\mathsf{v}\delta_1 - \mathsf{u}\delta_0),\\ &\partial_x Z(t,0) = 0, \quad \partial_x Z(t,1) = 0, \end{aligned} \tag{4.4}$$

and consequently (4.1) is turned into

$$\begin{aligned} &\partial_t h = \tfrac{1}{2}\Delta h + \tfrac{1}{2}|\nabla h|^2 + \xi - \tfrac{1}{2}(\mathsf{v}\delta_1 + \mathsf{u}\delta_0), \qquad t>0, x\in(0,1),\\ &\partial_x h(t,0) = 0, \quad \partial_x h(t,1) = 0. \end{aligned} \tag{4.5}$$

For the case of $\mathsf{u}=\mathsf{v}=0$, it is not hard to show that Brownian motion is invariant under (4.1), either by discrete method or the one presented in the previous section. The idea here is to absorb the boundary condition into the noise, in other words, we view (4.5) as (4.1) with homogeneous boundary conditions but the noise is changed into $\xi\mapsto\xi-\frac{1}{2}(\mathsf{v}\delta_1+\mathsf{u}\delta_0)$. The change of the noise is very singular and $-\frac{1}{2}(\mathsf{v}\delta_1+\mathsf{u}\delta_0)$ does not live in the Cameron-Martin space associated with ξ (which is L^2 space), so the law of $\xi-\frac{1}{2}(\mathsf{v}\delta_1+\mathsf{u}\delta_0)$ is not absolutely continuous with respect to ξ, but

[16] It turns out that there is an interesting extra 1/2 appearing in the Robin boundary condition for the multiplicative SHE due to non-smoothness of the solution but we will ignore it for the sake of discussion.

one may proceed by approximation so that with some smooth $\phi^\varepsilon_{\mathsf{u},\mathsf{v}} \to -\frac{1}{2}(\mathsf{v}\delta_1 + \mathsf{u}\delta_0)$ as $\varepsilon \to 0$, by Cameron-Martin theorem, there is a change of measure factor

$$\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t,\xi) = \exp\left(\langle \phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \xi\rangle_{L^2[0,t]\times[0,1]} - \tfrac{1}{2}t\|\phi^\varepsilon_{\mathsf{u},\mathsf{v}}\|^2_{L^2[0,1]}\right),$$

which is the Radon-Nikodym derivative of $\xi + \phi^\varepsilon_{\mathsf{u},\mathsf{v}}$ with respect to ξ. Suppose we consider

$$\begin{aligned} &\partial_t h^\varepsilon = \tfrac{1}{2}\Delta h^\varepsilon + \tfrac{1}{2}|\nabla h^\varepsilon|^2 + \xi + \phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \qquad t>0, x\in(0,1),\\ &\partial_x h^\varepsilon(t,0) = 0, \quad \partial_x h^\varepsilon(t,1) = 0. \end{aligned} \tag{4.6}$$

Then all the statistics we want to know about (4.6) can be computed using $\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}$. From now on, denote $\mathbf{h}$ the solution to (4.1) with $\mathsf{u} = \mathsf{v} = 0$. For any bounded functional F we have

$$\mathbf{E}F(h^\varepsilon_{\le t}) = \mathbf{E}F(\mathbf{h}_{\le t})\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t). \tag{4.7}$$

One may ask a very general question. Suppose we have a Markov process $\{\omega_s\}_s$ with an explicit invariant measure, and we perturb the dynamics, how does the invariant measure change? Of course "perturb" is a very vague word and there is no way to answer the question in general. In our interested case here, however, the perturbation is to shift the noise ξ by $\phi^\varepsilon_{\mathsf{u},\mathsf{v}}$ which can be viewed as a change of measure. The following discussion applies to all cases where the perturbation can be interpreted in this way.

For $\mathbf{h}$, the solution to the open KPZ with $\mathsf{u} = \mathsf{v} = 0$, Brownian motion is invariant in the sense that if $\mathbf{h}(0,x) = W(x)$ where W is a standard BM with $W(0) = 0$, then for any $t > 0$, $\mathbf{h}(t,x) - \mathbf{h}(t,0) \overset{law}{=} W(x)$. Sometimes, to subtract $\mathbf{h}(t,0)$ in all the relevant discussions is a bit inconvenient, so we will work with the stochastic Burgers equation. Define

$$u^\varepsilon = \nabla h^\varepsilon, \qquad \mathbf{u} = \nabla \mathbf{h}.$$

The starting point for our analysis is (4.7). By taking a bounded functional $F : E \to \mathbb{R}$, where E is some state space for Burgers, we have for any $t > 0$

$$\mathbf{E}F(u^\varepsilon_t) = \mathbf{E}F(\mathbf{u}_t)\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t). \tag{4.8}$$

Since the invariant measure for $\mathbf{u}_t$ is a white noise, we start u^ε_0 and $\mathbf{u}_0$ with a white noise so that the $\mathbf{u}_t$ on the r.h.s. is at stationarity. In particular, the r.h.s. can be rewritten as

$$\mathbf{E}F(\mathbf{u}_t)\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t) = \mathbf{E}\Big[F(\mathbf{u}_t)\mathbf{E}[\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t)|\mathbf{u}_t]\Big].$$

Since $\mathbf{u}_t$ has the law of a white noise, one may view the r.h.s. as an explicit integration with respect to the white noise measure: on a physics level, this gives

$$\mathbf{E}F(u^\varepsilon_t) = \int F(u)\mathbf{E}[\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t)|\mathbf{u}_t = u]e^{-\frac{1}{2}\int_0^1 u(x)^2dx}du.$$

Since F is arbitrary bounded function, the above identity shows that the law of u^ε_t is absolutely continuous respect to the white noise, and more importantly, the density takes the form

$$\rho^\varepsilon_t(u) = \mathbf{E}[\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t)|\mathbf{u}_t = u]. \tag{4.9}$$

Thus, to study the law of u^ε, it reduces to analyzing the above conditional expectation. The meaning of (4.9) is sort of clear: $\rho^\varepsilon_t(u)$ measures how likely the process reaches the state u at time t, so one is only interested in those realizations of the noise to make this event occur. Those realizations of the noise live in the Cameron-Martin space and the likelihood are given by $\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t)$, and the r.h.s. of (4.9) is an average of such likelihoods.

A generalization of (4.9) is as follows. We assumed that at $t = 0$, the law of u^ε is the same as a white noise so the density takes the form (4.9). If instead we assume the initial distribution of u^ε is absolutely continuous with respect to the white noise, with Radon-Nikodym derivative ρ_0^ε, then the law of u^ε at time t is also absolutely continuous with respect to the white noise, and its Radon-Nikodym derivative takes the form

$$\rho_t^\varepsilon(u) = \mathbf{E}[\rho_0^\varepsilon(\mathbf{u}_0)\mathscr{Q}_{\mathsf{u},\mathsf{v}}^\varepsilon(t)|\mathbf{u}_t = u]. \tag{4.10}$$

Here $\mathbf{u}_0$ is sampled from the invariant measure which is a white noise.

The above formula can be written down in very general settings. The difficulty lies in the fact that the conditional expectation is not the "standard" one in the sense that it is conditioning on future: $\mathscr{Q}_{\mathsf{u},\mathsf{v}}^\varepsilon(t)$ and $\mathbf{u}_t$ both depend on the noise up to time t. This is where the time reversal come to play.

4.1.1. *Time reversal of KPZ/Burgers.* Assuming that u is at stationarity, in the whole line or periodic setting. Since it is a Markov process at stationarity, one may ask what is the time-reversed process. By the discussion on the generator in Section 3.1, we know that, at least formally, the generator takes the form $\mathcal{L} = \mathcal{S} + \mathcal{A}$ where $\mathcal{S}$ and $\mathcal{A}$ are symmetric and anti-symmetric with respect to the inner product $\langle F, G\rangle = \int F(u)G(u)e^{-\frac{1}{2}\int u_x^2 dx}dx$. So $\mathcal{L}^* = \mathcal{S}^* + \mathcal{A}^* = \mathcal{S} - \mathcal{A}$ is the generator of the time reversed process. By the calculation of the generator it is clear that if we change the sign in front of the nonlinear term and consider

$$\partial_t \hat{u} = \tfrac{1}{2}\Delta\hat{u} - \tfrac{1}{2}\nabla\hat{u}^2 + \nabla\hat{\xi},$$

where $\hat{\xi}$ is some spacetime white noise, the generator of $\{\hat{u}_t\}_t$ is given by $\mathcal{L}^* = \mathcal{S} - \mathcal{A}$. In other words, this formal calculation shows that the time reversal of stochastic Burgers at stationarity is another stochastic Burgers but with the opposite sign in front of the nonlinear term[17]. To prove it rigorously, one can use discrete approximations.

Here is a recap of basic Markov chain[18] which is related to the above discussion. Suppose the state space E of the Markov chain is countable and π is a probability measure on E, and P is the transition matrix. Assuming $\pi(x) > 0$ for all $x \in E$ (otherwise shrink the state space). Define the inner product on $\ell^2(\pi)$ by $\langle F, G\rangle = \sum_x F(x)G(x)\pi(x)$. Define P^* through $\pi(x)P(x,y) = \pi(y)P^*(y,x)$, then P^* is the transition matrix of the time reversed chain at stationarity and it is the adjoint of P under the inner product $\langle\cdot,\cdot\rangle$.

For the open KPZ with $\mathsf{u} = \mathsf{v} = 0$, the above discussion also holds. From now on, fix $t > 0$, define

$$\hat{\mathbf{u}}_s = \mathbf{u}_{t-s}, \qquad s \in [0, t],$$

one can show that there exists some other spacetime white noise $\hat{\xi}$ such that

$$\partial_s \hat{\mathbf{u}}_s = \tfrac{1}{2}\Delta\hat{\mathbf{u}}_s - \tfrac{1}{2}\nabla\hat{\mathbf{u}}_s^2 + \nabla\hat{\xi}. \tag{4.11}$$

One can compare the above equation with the one for $\mathbf{u}$ itself

$$\partial_s \mathbf{u}_s = \tfrac{1}{2}\Delta\mathbf{u}_s + \tfrac{1}{2}\nabla\mathbf{u}_s^2 + \nabla\xi. \tag{4.12}$$

[17]It is easy to see that $-\hat{u}$ solves the same equation as u, so sometimes it is phrased as "the stochastic Burgers equation at stationarity satisfied the skew symmetry in the sense that $\{-u_{-s}\}_s$ has the same law as $\{u_s\}_s$"

[18]Problem Session

The idea is to use the above two equations to express ξ in terms of $\hat{\mathbf{u}}$ and $\hat{\xi}$, so $\mathscr{Q}^{\varepsilon}_{\mathsf{u},\mathsf{v}}(t)$ is expressed in terms of $\hat{\mathbf{u}}$ and $\hat{\xi}$, and the conditional expectation

$$\rho^{\varepsilon}_t(u) = \mathbf{E}[\mathscr{Q}^{\varepsilon}_{\mathsf{u},\mathsf{v}}(t)|\mathbf{u}_t = u] = \mathbf{E}[\mathscr{Q}^{\varepsilon}_{\mathsf{u},\mathsf{v}}(t)|\hat{\mathbf{u}}_0 = u] \tag{4.13}$$

can be viewed as the "standard" conditional expectation with respect to the past.

The Burgers dynamics is not reversible, which can be seen either from the two equations satisfied by $\mathbf{u}, \hat{\mathbf{u}}$ or from the formal calculation that gives $\mathcal{L}^* \neq \mathcal{L}$. This irreversibility creates most of the difficulties in our analysis.

To illustrate the main ideas, we study a simple reversible example where everything can be done rather easily.

4.1.2. *OU process.* Consider the OU process

$$dX_t = -X_t dt + dB_t, \tag{4.14}$$

the invariant measure is given by the Gaussian density $\frac{1}{\sqrt{\pi}} e^{-x^2}$. We view it as an (unrealistically) toy version of the equation satisfied by $\mathbf{u}$[19].

Recall that we were trying to understand what happens to the white noise invariant measure once we perturb the dynamics of $\mathbf{u}$ by shifting the noise $\xi \mapsto \xi + \phi^{\varepsilon}_{\mathsf{u},\mathsf{v}}$. To mimic this scenario, we perturb (4.14) by shifting $B_t \mapsto B_t + \theta t$ for some $\theta \in \mathbb{R}$. Denote the process by X^{θ} which solves $dX^{\theta}_t = -X^{\theta}_t dt + dB_t + \theta dt$. Of course we already know the invariant measure for the perturbed process is $\frac{1}{\sqrt{\pi}} e^{-(x-\theta)^2}$. In the following we will come up with a more complicated approach to derive this well-known fact and this approach will guide us to solve the open KPZ problem.

Similar to (4.8), take any bounded F, we have

$$\mathbf{E}F(X^{\theta}_t) = \mathbf{E}F(X_t) e^{\theta B_t - \frac{1}{2}\theta^2 t}.$$

Assuming X^{θ}_t starts from the invariant measure of X_t, so X_t is invariant on the r.h.s. By taking the conditional expectation, we know that the law of X^{θ}_t is absolutely continuous to the Gaussian density e^{-x^2} with the Radon-Nikodym derivative given by

$$\rho^{\theta}_t(x) = \mathbf{E}[e^{\theta B_t - \frac{1}{2}\theta^2 t} | X_t = x],$$

which is an analogue of (4.9).

Define the time reversed process $\hat{X}_s = X_{t-s}$, since the OU process at stationarity is reversible, we know that there exists a standard BM $\{\hat{B}_s\}_s$ adapted to the filtration associated with $\hat{X}$ such that

$$d\hat{X}_s = -\hat{X}_s ds + d\hat{B}_s. \tag{4.15}$$

Combining (4.14) and (4.15) (write in integral form and subtract one from the other), we have

$$B_t = \hat{B}_t - 2\hat{X}_t + 2\hat{X}_0, \tag{4.16}$$

[19]In the finite dimensions, there is the natural Lebesgue measure, so when referring to density, it is usually the density with respect to the Lebesgue measure. In infinite dimensional settings, we choose the white noise Gaussian measure as the base measure, so the density is with respect to the Gaussian measure. In the discussion here, one should view ρ^{θ}_t as the density with respect to the Gaussian measure.

which implies

$$\rho_t^\theta(x) = \mathbf{E}\Big[e^{\theta(\hat{B}_t - 2\hat{X}_t + 2\hat{X}_0) - \frac{1}{2}\theta^2 t} | \hat{X}_0 = x\Big]$$
$$= e^{2\theta x}\mathbf{E}\Big[e^{\theta(\hat{B}_t - 2\hat{X}_t) - \frac{1}{2}\theta^2 t} | \hat{X}_0 = x\Big].$$

At this stage one should be convinced that the above conditional expectation will converge to some constant c_θ that does not depend on x since $\hat{X}_t$ converges to invariant distribution as $t \to \infty$. Thus, $\rho_t^\theta(x) \to c_\theta e^{2\theta x}$, and this shows that the invariant measure of X^θ is absolutely continuous with respect to e^{-x^2} with Radon-Nikodym derivative given by $c_\theta e^{2\theta x}$. It is easy to compute what c_θ should be but we do not bother and conclude that the invariant measure has a density that is proportional to $e^{2\theta x}e^{-x^2}$, which is proportional to $e^{-(x-\theta)^2}$.

One can also use (4.13). Suppose we do not start from the invariant measure of X, but start from a measure that is absolutely continuous with respect to $\frac{1}{\sqrt{\pi}}e^{-x^2}$ with density $e^{2\theta x}$ (and pretend we did not know it is invariant for X^θ). Then, by (4.13), the law of X_t^θ is absolutely continuous with respect to $\frac{1}{\sqrt{\pi}}e^{-x^2}$ with density

$$\rho_t^\theta(x) = \mathbf{E}[e^{2\theta X_0}e^{\theta B_t - \frac{1}{2}\theta^2 t} | X_t = x].$$

Replacing B_t by $\hat{X}$ and $\hat{B}$, we have

$$\rho_t^\theta(x) = \mathbf{E}\Big[e^{2\theta\hat{X}_t}e^{\theta(\hat{B}_t - 2\hat{X}_t + 2\hat{X}_0) - \frac{1}{2}\theta^2 t} | \hat{X}_0 = x\Big] = e^{2\theta x}.$$

Thus the density $e^{2\theta x}$ is invariant.

The goal is to do (exactly) the same for the open KPZ equation. The obvious difference is that $X, \hat{X}$ solve the same type equation but $\mathbf{u}, \hat{\mathbf{u}}$ solve different equations. If we subtract (4.11) from (4.12), we end up with the nonlinear term.

4.2. **Regularity structures.** Recall that

$$\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t,\xi) = \exp\big(\langle\phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \xi\rangle_{L^2[0,t]\times[0,1]} - \tfrac{1}{2}t\|\phi^\varepsilon_{\mathsf{u},\mathsf{v}}\|^2_{L^2[0,1]}\big)$$

is the change of measure factor, and the goal is to express ξ in terms of the backward solution and noise. Instead of (4.8) and (4.11), we write the corresponding KPZ equation

$$\partial_s\hat{\mathbf{h}}_s = \tfrac{1}{2}\Delta\hat{\mathbf{h}}_s - \tfrac{1}{2}|\nabla\hat{\mathbf{h}}_s|^2 + \hat{\xi}$$
$$\partial_s\mathbf{h}_s = \tfrac{1}{2}\Delta\mathbf{h}_s + \tfrac{1}{2}|\nabla\mathbf{h}_s|^2 + \xi,$$

from which we derive

$$\langle\phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \xi\rangle_{L^2[0,t]\times[0,1]} = \langle\phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \hat{\xi}\rangle_{L^2[0,t]\times[0,1]} - \langle|\nabla\hat{\mathbf{h}}|^2, \phi^\varepsilon_{\mathsf{u},\mathsf{v}}\rangle_{L^2[0,t]\times[0,1]}$$
$$+ 2\langle\phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \hat{\mathbf{h}}_0\rangle_{L^2[0,1]} - 2\langle\phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \hat{\mathbf{h}}_t\rangle_{L^2[0,1]}.$$

Compare to (4.16), there is the extra nonlinear term which comes from the non-reversibility of the dynamics. As a result

$$\begin{aligned}\mathscr{Q}^\varepsilon_{\mathsf{u},\mathsf{v}}(t,\xi) &= \exp\big(\langle\phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \hat{\xi}\rangle_{L^2[0,t]\times[0,1]} - \tfrac{1}{2}t\|\phi^\varepsilon_{\mathsf{u},\mathsf{v}}\|^2_{L^2[0,1]}\big)\\ &\times \exp(-\langle|\nabla\hat{\mathbf{h}}|^2, \phi^\varepsilon_{\mathsf{u},\mathsf{v}}\rangle_{L^2[0,t]\times[0,1]})\\ &\times \exp(2\langle\phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \hat{\mathbf{h}}_0\rangle_{L^2[0,1]} - 2\langle\phi^\varepsilon_{\mathsf{u},\mathsf{v}}, \hat{\mathbf{h}}_t\rangle_{L^2[0,1]}).\end{aligned} \tag{4.17}$$

To show the invariance of the measure tilted by (4.2), i.e., if we start from the density Q then at any $t > 0$, we have the density Q, according to (4.10) it is enough

to show that for any $t > 0$,

$$\begin{aligned} Q(u) &= \lim_{\varepsilon} \mathbf{E}[Q(\mathbf{u}_0)\mathscr{Q}^{\varepsilon}_{\mathsf{u},\mathsf{v}}(t,\xi)|\mathbf{u}_t = u] \\ &= \lim_{\varepsilon} \mathbf{E}[Q(\hat{\mathbf{u}}_t)\mathscr{Q}^{\varepsilon}_{\mathsf{u},\mathsf{v}}(t,\xi)|\hat{\mathbf{u}}_0 = u]. \end{aligned} \tag{4.18}$$

Here we have abused notation to write $Q(u)$ which equals to the $Q(h)$ in (4.2) with $h(x) = \int_0^x u(y)dy$. To pass to the limit of $\varepsilon \to 0$ in the expression of $\mathscr{Q}^{\varepsilon}_{\mathsf{u},\mathsf{v}}$ in (4.17), the first line is a change of measure factor, and the last line converges since $\hat{\mathbf{h}}$ is continuous and $\phi^{\varepsilon}_{\mathsf{u},\mathsf{v}} \to -\frac{1}{2}(\mathsf{v}\delta_1 + \mathsf{u}\delta_0)$. Thus, the key difficulty is the second line

$$\exp(-\langle|\nabla\hat{\mathbf{h}}|^2, \phi^{\varepsilon}_{\mathsf{u},\mathsf{v}}\rangle_{L^2[0,t]\times[0,1]}).$$

Most of the pages in [13] were spent on showing that the weak convergence

$$\langle|\nabla\hat{\mathbf{h}}|^2, \phi^{\varepsilon}_{\mathsf{u},\mathsf{v}}\rangle_{L^2[0,t]\times[0,1]} \Rightarrow N(\mu t, \sigma^2 t) \tag{4.19}$$

for some explicit μ, σ^2 (depending on u, v), and the limiting Gaussian is independent of everything else. As a result, this term is averaged out explicitly when passing to the limit in (4.18). On the other hand, (4.18) resembles some martingale condition on some functional related to Q. Indeed, after passing to the limit and removing the nonlinear term, one can show the rest is a martingale in the backward filtration and complete the proof. We will not go into the detail here. More heuristic discussion can be found in [13, Section 1.1].

The proof of (4.19) is through the theory of regularity structures [19], which we will not discuss at all in these notes. But it is worth mentioning that this machinery is not used here to make sense of the singular SPDE – there is the Hopf-Cole and a well-defined solution theory. Instead we use the reconstruction theorem to justify that only a local expansion of $|\nabla\hat{\mathbf{h}}|^2$ contributes to the limit in (4.19).

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