

21-425 Probability and Martingales: Midterm 1.

2026-09-16

- You have 50 minutes. The exam has a total of 4 questions and 40 points.
- You may not use phones, calculators, or other electronic devices. You may not give or receive assistance.
- This is a closed book test. You may not use notes, study sheets, or online resources.
- You may freely use results from class / homework, except when you are asked to prove them. In some cases the question you're asked may be a simplification, a step, or a special case of the more general result proved. For these questions you should not trivialize the problem by using an almost identical (or more general) result from class/homework, and instead provide a complete proof here. You may, of course, use other results from class / homework that are independent of the question at hand.
- You do not need to turn in scratch work. Good luck.

Survey question: How helpful was AI in this course? ☐ 1 (not much) ☐ 2 ☐ 3 ☐ 4 ☐ 5 (a lot)

1. Let X be a random variable, and let $F_X(x) = \mathbf{P}(X \leq x)$ be the CDF of X .

☐ 5 (a) For every $x \in \mathbb{R}$, must $\lim_{n \rightarrow \infty} F(x - 1/n) = F(x)$? Prove it, or find a counter example.

☐ 5 (b) For every $x \in \mathbb{R}$, must $\lim_{n \rightarrow \infty} F(x + 1/n) = F(x)$? Prove it, or find a counter example.

☐ 10 2. Suppose X, Y are two discrete random variables such that both $\mathbf{E}X$ and $\mathbf{E}Y$ are defined (i.e. the sum defining the expectation converges absolutely). Show that $\mathbf{E}(X + Y) = \mathbf{E}X + \mathbf{E}Y$.

☐ 10 3. Let $(p_N)_{N \in \mathbb{N}}$ be a sequence in $(0, 1)$, and for every $N \in \mathbb{N}$ suppose $X_N \sim \text{Binom}(N, p_N)$. If for some $\lambda > 0$, the sequence $Np_N \rightarrow \lambda$ as $N \rightarrow \infty$, then show that

$$\lim_{N \rightarrow \infty} \mathbf{P}(X_N = k) = \mathbf{P}(Y = k) \quad \text{for every } k \in \mathbb{N}.$$

where $Y \sim \text{Pois}(\lambda)$.

4. Let $p \in (0, 1)$ and suppose $X \sim \text{Geom}(p)$.

☐ 5 (a) Compute $\mathbf{E}X$ in terms of p .

☐ 5 (b) Compute $\text{Var}(X)$ in terms of p .