21-269 Vector Analysis: Midterm 1.

2025-09-26

- *You have 50 minutes. The exam has a total of 4 questions and 40 points.*
- *You may not use phones, calculators, or other electronic devices. You may not give or receive assistance.*
- *This is a closed book test. You may not use notes, study sheets, or online resources.*
- *You may freely use results from class / homework, except when you are asked to prove them. In some cases the question you're asked may be a simplification, a step, or a special case of the more general result proved. For these questions you should not trivialize the problem by using an almost identical (or more general) result from class/homework, and instead provide a complete proof here. You may, of course, use other results from class / homework that are independent of the question at hand.*
- *If you need more room for an answer, write your name on the extra sheet and insert it in your exam.*
- *You do not need to turn in scratch work. Good luck.*

Survey question: How helpful was AI in this course? ◯ 1 (not much) ◯ 2 ◯ 3 ◯ 4 ◯ 5 (a lot)

Notation: In this exam, X, Y denote arbitrary metric spaces with distance function d. The spaces $\mathbb{R}$ or $\mathbb{R}^d$ are metric spaces with the standard Euclidean metric.

[10] 1. Suppose $a \in X$ and $g \colon X \to \mathbb{R}$ is a function such that $\lim_{x \to a} g(x) = \ell \neq 0$. Show that $\lim_{x \to a} 1/g(x) = 1/\ell$.

[10] 2. Let $f \colon X \to Y$ be a function, and $a \in X$. Show that f is continuous at a if and only if for every sequence $(a_n)_{n \in \mathbb{N}}$ in X such that $a_n \xrightarrow{n \to \infty} a$ we have $f(a_n) \xrightarrow{n \to \infty} f(a)$.

[10] 3. Let $a \in \mathbb{R}^d$, and $C \subseteq \mathbb{R}^d$ be a non-empty closed set. Must there exist $x_* \in C$ such that

$$|x_* - a| = \inf\{|x - a| \mid x \in C\}\,?$$

Prove / disprove it.

[10] 4. Consider a collection of sets $\{C_n \mid n \in \mathbb{N}\}$ with the following properties:

1. For every $n \in \mathbb{N}$, $C_n \subseteq \mathbb{R}^d$ is closed, non-empty, and $C_{n+1} \subseteq C_n$.
2. There exists $R > 0$ such that $C_1 \subseteq B(0, R)$.

Can the intersection $\bigcap_{n=1}^{\infty} C_n$ be empty? Prove / disprove it.