21-326 Markov Chains: Midterm 2.

2025-11-05

- *You have 50 minutes. The exam has a total of 4 questions and 40 points.*
- *You may not use phones, calculators, or other electronic devices. You may not give or receive assistance.*
- *This is a closed book test, however, you are permitted to bring in one sheet of paper (US Letter paper sized) to the exam, on which you may may* handwrite *anything you wish* on one side only. *No photocopies, and no typed printouts. You must turn in this sheet with your exam.*
- *In general, you may use (without proof) results from class / homework. However, some questions may ask you to prove a result that is very similar to one from class / homework; or a step used in the proof of a result from class / homework. For these questions you should not trivialize the problem by saying "done in class", and instead provide a complete proof here. (You may, of course, use other results from class / homework provided they are independent of the question that was asked.) Good luck.*

[10] 1. Let π be a probability distribution on $\mathcal{X}$, and $A \subseteq \mathcal{X}$ be an event we want the estimate the probability of. Let X_1, ..., X_N be i.i.d. samples from π, and define

$$\hat{\pi}_N(A) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_A(X_n) \quad \text{and} \quad \mathcal{E}_N = \frac{1}{\pi(A)} \sqrt{\boldsymbol{E}(\hat{\pi}_N(A) - \pi(A))^2}\,.$$

Here $\pi(A) = \sum_{a \in A} \pi(a)$, $\hat{\pi}_N(A)$ is an estimator for $\pi(A)$, and $\mathcal{E}_N$ is a measure of the *relative error*. Given $\varepsilon > 0$ find N (in terms of $\pi(A)$ and ε) so that that we necessarily have $\mathcal{E}_N < \varepsilon$.

[10] 2. Let V, S be non-empty finite sets, let $\mathcal{X} = \{x \colon V \to S\}$, and let $\pi \colon \mathcal{X} \to (0, \infty)$ be a probability distribution on $\mathcal{X}$. Given let $x \in \mathcal{X}$, $v \in V$ and define $\mathcal{X}(x, v) = \{y \in \mathcal{X} \mid y(w) = x(w)\ \forall w \neq v\}$. Also define $\pi(\mathcal{X}(x, v)) = \sum_{y \in \mathcal{X}(x,v)} \pi(y)$. The *Glauber dynamics* is a Markov chain on $\mathcal{X}$ that is defined as follows. Given $X_n = x$, choose $v \in V$ uniformly at random. Then chose $y \in \mathcal{X}(x, v)$ with probability $\pi(y)/\pi(\mathcal{X}(x, v)$ and set $X_{n+1} = y$.

Show that this Glauber chain is reversible with stationary distribution π.

[10] 3. Let $N \geqslant 2$ be a natural number, $\mathcal{X} = \{0, \dots, N-1\}$ and consider the symmetric random walk which wraps around on the boundary. That is, given $X_n = x$, choose $a \in \{\pm 1\}$ independently, uniformly at random, and set $X_{n+1} = x + a \pmod N$. Given a complex number ζ, define the function $\varphi \colon \mathcal{X} \to \mathbb{C}$ by $\varphi(k) = \zeta^k$. For what values of ζ is φ an eigenfunction of the transition matrix? Assuming all eigenfunctions are of this form, find the absolute spectral gap of the transition matrix.

[10] 4. Let P be the transition matrix of an reversible, irreducible chain with stationary distribution π. Find $\alpha \in \mathbb{R}$ so that for every function $f \colon \mathcal{X} \to \mathbb{R}$, we have $\langle Pf, f \rangle_\pi \geqslant \alpha \langle f, f \rangle_\pi$. (Prove your α works for every function f.)

NOTE: Here $\langle f, g \rangle_\pi$ is the inner-product defined by $\langle f, g \rangle_\pi = \sum_{x \in \mathcal{X}} f(x) g(x) \pi(x)$, and the state space $\mathcal{X}$ is a non-empty finite set.