21-326 Markov Chains: Midterm 1.

2025-09-26

- *You have 50 minutes. The exam has a total of 4 questions and 40 points.*
- *You may not use phones, calculators, or other electronic devices. You may not give or receive assistance.*
- *This is a closed book test, however, you are permitted to bring in one sheet of paper (US Letter paper sized) to the exam, on which you may may* handwrite *anything you wish* on one side only. *No photocopies, and no typed printouts. You must turn in this sheet with your exam.*
- *In general, you may use (without proof) results from class / homework. However, some questions ask you to prove a result that is very similar to one from class / homework. For these questions you should not trivialize the problem by saying "done in class", and instead provide a complete proof here. (You may, of course, use other results from class / homework provided they are independent of the question that was asked.) Good luck.*

In this exam (X_n) denotes a time homogeneous Markov chain with transition matrix P on a finite state space $\mathcal{X}$.

$\boxed{10}$ 1. True or false: For every $x_1 \in \mathcal{X}$, conditioned on the event $\{X_1 = x_1\}$, the random variables X_2 and X_0 are (conditionally) independent. Prove it, or find a counter example.

$\boxed{10}$ 2. Let $S \subseteq \mathcal{X}$ be non-empty, define $\tau = \min\{n \geqslant 0 | X_n \in S\}$, and suppose $\boldsymbol{P}(\tau < \infty) = 1$. Given a function $f \colon \mathcal{S} \to \mathbb{R}$, define

$$u(x) \stackrel{\text{def}}{=} \boldsymbol{E}^x f(X_\tau) = \boldsymbol{E}(f(X_\tau) \mid X_0 = x)\,.$$

Show that

$$\begin{aligned} (I - P)u &= 0 \qquad \text{in } \mathcal{X} - S\,, \\ u &= f \qquad \text{in } S\,. \end{aligned}$$

Recall Pu is the function defined by $Pu(x) = \sum_{y \in \mathcal{X}} P(x,y)u(y)$, and I is the identity matrix.

$\boxed{10}$ 3. Consider the lazy nearest neighbor random walk on $\mathbb{Z}$, whose transition matrix is given by $P(m,n) = 1/3$ if $n - m \in \{0, \pm 1\}$ and $P(m,n) = 0$ otherwise. Does this Markov chain have a stationary distribution? Find it, or prove it doesn't exist.

$\boxed{10}$ 4. Let $N \in \mathbb{N}$, $N \geqslant 3$, $\mathcal{X} = \{0, 1, \dots, N-1\}$, and consider a Markov chain on $\mathcal{X}$ whose transition matrix satisfies

$$P(0,0) = 1\,, \quad P(N-1, 0) = P(N-1, N-2) = \frac{1}{2} \quad \text{and} \quad P(k, k \pm 1) = \frac{1}{2}\,,$$

for every $k \in \{1, \dots, N-2\}$. Let μ_0 be an arbitrary probability distribution on $\mathcal{X}$, and for every $n \in \mathbb{N}$ define $\mu_n = \mu_0 P^n$.

True or false: There exists a probability distribution π such that $\|\mu_n - \pi\|_{\mathrm{TV}} \to 0$ as $n \to \infty$. Prove it, or find a counter example.