21-326 Markov Chains: Final.

2025-12-14

10 1. Let (X_n) be a time homogeneous Markov chain finite state space $\mathcal{X}$. True or false: For any $N \in \mathbb{N}$, $N \geqslant 1$, and any sets $A_1, A_2 \subseteq \mathcal{X}$, and any $x, y \in \mathcal{X}$ we have

$$\boldsymbol{P}(X_{N+2} \in A_2, X_{N+1} \in A_1 \mid X_N = x, X_{N-1} = y) = \boldsymbol{P}(X_2 \in A_2, X_1 \in A_1 \mid X_0 = x)\,.$$

Prove it, or find a counter example. (Suggestion: Express both sides in terms of the transition matrix.)

10 2. A gambler plays a game where they win \$1 with probability 1/2, and lose \$1 with probability 1/2. They start with a fortune of \$$k$. They play this repeatedly until they make \$$N$, or lose all their money. Find the expected number of times they play this game as a function of N and the initial fortune k.

10 3. Let $\mathcal{X}$ be a finite set, and $p, q \colon \mathcal{X} \to (0, 1]$ be a two probability distributions. Let X_n be i.i.d. random variables with common distribution p. Let U_n be independent, Unif$([0, 1])$ distributed random variables. Define

$$N = \min\Big\{ n \;\Big|\; U_n \leqslant \frac{q(X_n)}{M p(X_n)} \Big\} \quad \text{where} \quad M = \max_{x \in \mathcal{X}} \Big\{ \frac{q(x)}{p(x)} \Big\}\,,$$

and set $Y = X_N$. Show that the distribution of Y is q.

10 4. Let (X_n) be an irreducible, aperiodic, time homogeneous Markov chain with on a finite state space, and let $t_{mix}(\varepsilon)$ denote the ε-mixing time of the chain. Show that for any $0 < \delta < \varepsilon < 1/2$ we have

$$t_{mix}(\delta) \leqslant t_{mix}(\varepsilon) \log_{2\varepsilon} \delta$$

10 5. Let (X_n) be a time homogeneous Markov chain with a transition matrix P on a finite state space. Let $S \subseteq \mathcal{X}$, and define $\tau = \min\{n \geqslant 0 \mid X_n \in S\}$. Suppose there exists $\delta > 0$, $N \in \mathbb{N}$ such that for every $x \in \mathcal{X}$ we have $\boldsymbol{P}^x(\tau_S \leqslant N) \geqslant \delta$. Show that for any initial distribution μ, we have $\boldsymbol{P}^\mu(\tau_S > 2N) \leqslant (1 - \delta)^2$. (Recall for any event A, $\boldsymbol{P}^\mu(A)$ denotes the probability of the event A given that $X_0 \sim \mu$.)

10 6. Let (X_n) be a Markov chain on a finite state space $\mathcal{X}$ with an irreducible, reversible transition matrix P. Let π be the stationary distribution, and γ^* be the absolute spectral gap. For any function $f \colon \mathcal{X} \to \mathbb{R}$, show that

$$\|P^n f - \mathcal{I}_\pi f\|_{\ell^2(\pi)} \leqslant (1 - \gamma_*)^n \|f - \mathcal{I}_\pi f\|_{\ell^2(\pi)}\,, \quad \text{where} \quad \mathcal{I}_\pi f = \sum_{x \in \mathcal{X}} f(x) \pi(x)\,.$$

NOTE: We proved this in class. Feel free to use without proof the lemmas we proved leading up to this, that were proved independently of this statement. (If your notation is very different from what we had in class, please explain it.)

10 7. Let (X_n) be a time homogeneous Markov chain with an irreducible transition matrix P on a finite state space. Let (Y_n) be the associated lazy chain whose transition matrix is $Q = (I + P)/2$, where I is the identity matrix. Let $S \subseteq \mathcal{X}$ and define $\sigma = \min\{n \mid X_n \in S\}$, $\tau = \min\{n \mid Y_n \in S\}$. (Suppose further both σ and τ are finite almost surely.) Given a function $g \colon S^c \to \mathbb{R}$ define the functions $u, v \colon \mathcal{X} \to \mathbb{R}$ by

$$u(x) = \boldsymbol{E}^x \sum_{n=0}^{\sigma-1} g(X_n) \quad \text{and} \quad v(x) = \boldsymbol{E}^x \sum_{n=0}^{\tau-1} g(Y_n)\,.$$

Find a formula expressing v in terms of u and prove it.

NOTE: Here we use the convention that $\sum_{n=0}^{-1} g(X_n) = 0$, and so $u(x) = v(x) = 0$ for $x \in S$.

10 8. Show that the mixing time of the lazy random walk on the N cycle is at most N^2. Explicitly, let $N \in \mathbb{N}$, $N \geqslant 2$, and define $\mathcal{X} = \{0, \dots, N-1\}$. Let P be a transition matrix on $\mathcal{X}$ defined by $P(x, y) = 1/2$ if $x = y$, $P(x, y) = 1/4$ if $x - y = \pm 1 \pmod N$, and $P(x, y) = 0$ otherwise. Let (X_n) be a time homogeneous Markov chain on $\mathcal{X}$ with transition matrix P. Show that the mixing time of this chain is at most N^2. (Here mixing time is $t_{mix}(1/4)$.)