Generalizations of the Borsuk–Ulam theorem

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(joint work with Henry Adams and Johnathan Bush)

The Borsuk–Ulam theorem states that any odd map $f\colon S^n \to \mathbb{R}^n$, that is, a map satisfying $f(-x) = -f(x)$ for all $x \in S^n$, has a zero. This result has found numerous applications across mathematics—many of them in combinatorics and discrete geometry, such as Lovász' proof of Kneser's conjecture or the Ham–Sandwich theorem. In these applications one parametrizes (a subspace of) possible solutions by the n-sphere S^n in a symmetric way and then constructs the map f as measuring n equations that commute with the antipodal symmetry. Invoking the Borsuk–Ulam theorem then finishes the proofs.

If fewer equations need to be balanced, and thus f is a map to $\mathbb{R}^k$ with $k < n$, we expect more solutions. Several suitable such strengthenings of the Borsuk–Ulam theorem exist. For example, it is a consequence of a result of Yang [5] that any odd map $f\colon S^n \to \mathbb{R}^k$ has an $(n-k)$-dimensional subspace of zeros.

Here we address the question, what can be shown if the system of equations is overdetermined, for an odd map $f\colon S^n \to \mathbb{R}^k$ with $k > n$? Certainly a generic map f will not have any zeros. In joint work with Henry Adams and Johnathan Bush [2], we prove:

Theorem 1. *(a) If $f\colon S^{2n-1} \to \mathbb{R}^{2kn+2n-1}$ is odd and continuous, then there is a subset $X \subset S^{2n-1}$ of diameter at most $\frac{2\pi k}{2k+1}$ such that $\operatorname{conv}(f(X))$ contains the origin.*

(b) If $f\colon S^n \to \mathbb{R}^{n+2}$ is odd and continuous, then there is a subset $X \subset S^n$ of diameter at most the diameter of the regular $(n+1)$-simplex inscribed in S^n such that $\operatorname{conv}(f(X))$ contains the origin.

Here S^n carries the instrinsic metric, where each closed geodesic has length 2π. Part (a) for $k = 0$ is the usual Borsuk–Ulam theorem for maps from odd-dimensional spheres. It should be emphasized that the proof of Theorem 1 uses the Borsuk–Ulam theorem. The new ingredient of [1, 2] is a lower bound for the topology of a certain configuration space of nearby points: Given a metric space X and scale parameter $\varepsilon > 0$, the metric thickening X_ε is the space of all probability measures in X with finite support of diameter less than ε. We equip this space with the 1-Wasserstein metric, or metric of optimal transport. Lower bounding the homotopical connectivity of the metric thickening of spheres at varying scale parameters ε, together with the classical Borsuk–Ulam theorem, yields Theorem 1.

Here we outline one application of this result. A trigonometric polynomial is an expression of the form $p(t) = c + \sum_{k=1}^n a_k \cos(kt) + b_k \sin(kt)$, inducing a map $S^1 \to \mathbb{R}$. In the case that $c = 0$, we call p a *homogeneous* trigonometric polynomial. The set $S \subset \{1, \dots, n\}$ of integers k with $a_k \neq 0$ or $b_k \neq 0$ is called the *spectrum* of p, and the largest integer in S is the *degree* of p. The spectrum of p constrains the set of roots of p; for example, if p is homogeneous of degree n then it has a root on any closed circular arc of length $\frac{2\pi n}{n+1}$; see [3, 4].

If the spectrum of p consists only of odd integers, then p is called a *raked* trigonometric polynomial. We show the following structural result about the roots of raked trigonometric polynomials:

Theorem 2. *Let $X \subset S^1$ be such that $\mathrm{diam}(X) < \frac{2\pi k}{2k+1}$. Then there is a raked homogeneous trigonometric polynomial of degree $2k-1$ that is positive on X. Moreover, there is a set $X \subset S^1$ of diameter $\frac{2\pi k}{2k+1}$ such that no raked homogeneous trigonometric polynomial of degree $2k-1$ is positive on X.*

Apply part (a) of Theorem 1 for $n = 1$ to the symmetric trigonometric moment curve

$$\gamma\colon S^1 \to \mathbb{R}^{2k}, \quad t \mapsto (\sin(t), \cos(t), \sin(3t), \cos(3t), \dots, \cos((2k-1)t).$$

There is a set $X \subset S^1$ of diameter at most $\frac{2\pi k}{2k+1}$ such that the convex hull of $\gamma(X)$ captures the origin. In particular, no hyperplane can separate $\gamma(X)$ from the origin and thus the inner product $\langle z, \gamma(X)\rangle$ has to change sign for every $z \in \mathbb{R}^{2k} \setminus \{0\}$. The inner products of a non-zero vector z with γ range over all non-zero raked homogeneous trigonometric polynomials of degree at most $2k-1$, which proves the second part of Theorem 2.

References

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