Countable Borel Equivalence Relations II

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A quick recap

Theorem (Feldman-Moore)

If E is a countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.

Warning

- The proof of the Feldman-Moore Theorem does not produce a "canonical group action".
- It is sometimes difficult to express a countable Borel equivalence relation as the orbit equivalence relation arising from a "natural action." Cf. the Turing equivalence relation.

Comparing orbit equivalence relations

Stating the obvious

If G, H are countable groups and X, Y are a standard Borel G-space, H-space respectively, then the following are equivalent:

- $E_G^X \leq_B E_H^Y$.
- *There exist a Borel map $f : X \to Y$ such that for all $a, b \in X$,*

$$G \cdot a = G \cdot b \iff H \cdot f(a) = H \cdot f(b).$$

The Fundamental Question

- *Does the complexity of E_G^X reflect the structural complexity of the group G?*
- *To what extent does the data (X, E_G^X) "remember" G and its action on X?*

An easy counterexample ...

- For each countable group G, consider the Borel action of G on $G \times [\,0, 1\,]$ defined by $g \cdot (h, r) = (gh, r)$.
- Then the Borel map $(h, r) \mapsto (1_G, r)$ selects a point in each G-orbit, and so the corresponding orbit equivalence relation is smooth.

Observation

If G acts freely on X and preserves a probability measure, then E_G^X isn't smooth.

Definition

The Borel action of the countable group G on the standard Borel space X is free iff $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$.

Not so ad hoc

Theorem (Dougherty-Jackson-Kechris)

Let G be a countable group and let X be a standard Borel G-space. If X does not admit a G-invariant probability measure, then for every countable group $H \supseteq G$, there exists a Borel action of H on X such that $E_H^X = E_G^X$.

Theorem

If E is a countable aperiodic Borel equivalence relation, then E can be realised as the orbit equivalence relation of a faithful Borel action of uncountably many countable groups.

Definition

A countable Borel equivalence relation E is aperiodic iff every E-class is infinite.

The obvious question

Question

Let E be a nonsmooth countable Borel equivalence relation. Does there necessarily exist a countable group G with a free measure-preserving Borel action on a standard probability space (X, μ) such that $E \sim_B E_G^X$?

Easy Observation

Suppose that E is a countable Borel equivalence relation on an uncountable standard Borel space. Then there exists a countable group G and a standard Borel G-space X such that:

- G preserves a nonatomic probability measure μ on X.
- $E \sim_B E_G^X$.

Free actions

Definition

- *The Borel action of the countable group G on the standard Borel space X is free iff $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$.*
  In this case, we say that X is a free standard Borel G-space.
- *The countable Borel equivalence relation E on X is free iff there exists a countable group G with a free Borel action on X such that $E_G^X = E$.*
- *The countable Borel equivalence relation E is essentially free iff there exists a free countable Borel equivalence relation F such that $E \sim_B F$.*

Question (Jackson-Kechris-Louveau)

Is every countable Borel equivalence relation essentially free?

Some closure properties

Theorem (Jackson-Kechris-Louveau)

Let E, F be countable Borel equivalence relations on the standard Borel spaces X, Y respectively.

- *If $E \leq_B F$ and F is essentially free, then so is E.*
- *If $E \subseteq F$ and F is essentially free, then so is E.*

Corollary

The following statements are equivalent:

- *Every countable Borel equivalence relation is essentially free.*
- *E_∞ is essentially free.*

Essentially free countable Borel equivalence relations

Theorem (S.T. 2006)

The class of essentially free countable Borel equivalence relations does not admit a universal element. In particular, E_∞ is not essentially free.

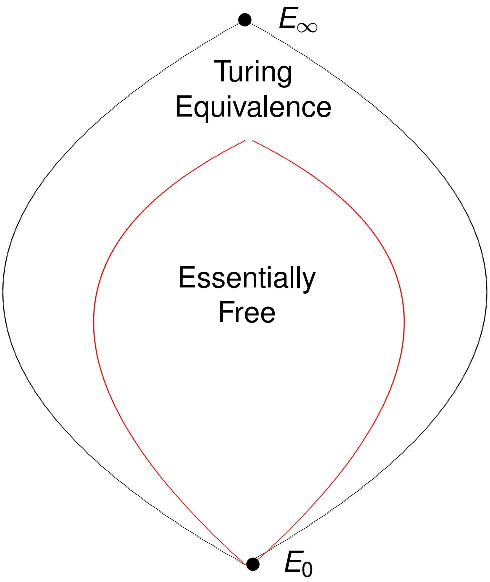
Corollary

$\equiv_T$ is not essentially free.

Proof.

Identifying the free group $\mathbb{F}_2$ with a suitably chosen group of recursive permutations of $\mathbb{N}$, we have that $E_\infty \subseteq \equiv_T$. □

A map of the world



Bernoulli actions

- Let G be a countably infinite group and consider the shift action on $\mathcal{P}(G) = 2^G$.
- Then the usual product probability measure μ on 2^G is G-invariant and the free part of the action

$$\mathcal{P}^*(G) = (2)^G = \{x \in 2^G \mid g \cdot x \neq x \text{ for all } 1 \neq g \in G\}$$

  has μ-measure 1.
- Let E_G be the corresponding orbit equivalence relation on $(2)^G$.

Observation

If $G \leqslant H$, then $E_G \leq_B E_H$.

Proof.

The inclusion map $\mathcal{P}^*(G) \hookrightarrow \mathcal{P}^*(H)$ is a Borel reduction from E_G to E_H. □

Homomorphisms

Definition

- *Let E be a countable Borel equivalence relation on the standard Borel space X with invariant probability measure μ.*
- *Let F be a countable Borel equivalence relation on the standard Borel space Y.*
- *Then the Borel homomorphism $f : X \to Y$ from E to F is said to be μ-trivial iff there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that f maps Z into a single F-class.*

Definition

If G, H are countable groups, then the group homomorphism $\pi : G \to H$ is a virtual embedding iff $|\ker \pi| < \infty$.

An easy consequence of Popa superrigidity

Theorem

- *Let $G = SL_3(\mathbb{Z}) \times S$, where S is any countable group.*
- *Let H be any countable group and let Y be a free standard Borel H-space.*

If there exists a μ-nontrivial Borel homomorphism from E_G to E_H^Y, then there exists a virtual embedding $\pi : G \to H$.

Remark

In particular, the conclusion holds if there exists a Borel subset $Z \subseteq (2)^G$ with $\mu(Z) = 1$ such that $E_G \restriction Z \leq_B E_H^Y$.

Essentially free countable Borel equivalence relations

Theorem

If E is an essentially free countable Borel equivalence relation, then there exists a countable group G such that $E_G \not\leq_B E$.

Corollary

The class of essentially free countable Borel equivalence relations does not admit a universal element. In particular, E_∞ is not essentially free.

Proof of Theorem

- We can suppose that $E = E_H^X$ is realised by a free Borel action on X of the countable group H.
- Let L be a finitely generated group which does not embed into H.
- Let $S = L * \mathbb{Z}$ and let $G = SL_3(\mathbb{Z}) \times S$.
- Then G has no finite normal subgroups and so there does not exist a virtual embedding $\pi : G \to H$.
- Hence $E_G \not\leq_B E_H^X$.

Uncountably many free countable Borel equivalence relations

Definition

- *For each prime* $p \in \mathbb{P}$, *let* $A_p = \bigoplus_{i=0}^{\infty} C_p$, *where* C_p *is the cyclic group of order* p.
- *For each subset* $S \subseteq \mathbb{P}$, *let*

$$G_S = SL_3(\mathbb{Z}) \times \bigoplus_{p \in S} A_p.$$

Theorem

If $S, T \subseteq \mathbb{P}$, *then* $E_{G_S} \leq_B E_{G_T}$ *iff* $S \subseteq T$.

Ergodicity

Definition

Let G be a countable group and let X be a standard Borel G-space. Then the G-invariant probability measure μ is said to be ergodic *iff $\mu(A) = 0, 1$ for every G-invariant Borel subset $A \subseteq X$.*

Example

Every countable group G acts ergodically on $(\, (2)^G, \mu\,)$.

Theorem

If μ is a G-invariant probability measure on the standard Borel G-space X, then the following statements are equivalent.

- *The action of G on (X, μ) is ergodic.*
- *If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction M$ is a constant function.*

Towards uncountably many non-essentially free countable Borel equivalence relations

Definition

The countable groups G, H are virtually isomorphic *iff there exist finite normal subgroups $N \trianglelefteq G$, $M \trianglelefteq H$ such that $G/N \cong H/M$.*

Lemma

There exists a Borel family $\{S_x \mid x \in 2^{\mathbb{N}}\}$ of f.g. groups such that if $G_x = SL_3(\mathbb{Z}) \times S_x$, then the following conditions hold:

- *If $x \neq y$, then G_x and G_y are not virtually isomorphic.*
- *If $x \neq y$, then G_x doesn't virtually embed in G_y.*

Definition

For each Borel subset $A \subseteq 2^{\mathbb{N}}$, let $E_A = \bigsqcup_{x \in A} E_{G_x}$.

Not essentially free

Lemma

If the Borel subset $A \subseteq 2^{\mathbb{N}}$ is uncountable, then E_A is not essentially free.

Proof.

- Suppose that $E_A \leq_B E_H^Y$, where H is a countable group and Y is a free standard Borel H-space.
- Then for each $x \in A$, we have that $E_{G_x} \leq_B E_H^Y$ and so there exists a virtual embedding $\pi_x : G_x \to H$.
- Since A is uncountable and each G_x is finitely generated, there exist $x \neq y \in A$ such that $\pi_x[G_x] = \pi_y[G_y]$.
- But then G_x, G_y are virtually isomorphic, which is a contradiction.

□

Uncountably many non-essentially free relations

Lemma

$E_A \leq_B E_B$ iff $A \subseteq B$.

Proof.

- Suppose that $E_A \leq_B E_B$.
- Suppose also that $A \not\subseteq B$ and that $x \in A \smallsetminus B$.
- Then there exists a Borel reduction from E_{G_x} to E_B

$$f : (2)^{G_x} \to \bigsqcup_{y \in B} (2)^{G_y}.$$

- By ergodicity, there exists μ_x-measure 1 subset of $(2)^{G_x}$ which maps to a fixed $(2)^{G_y}$.
- This yields a μ_x-nontrivial Borel homomorphism from E_{G_x} to E_{G_y} and so G_x virtually embeds into G_y, which is a contradiction.