Countable Borel Equivalence Relations I

Simon Thomas

Rutgers University

17th November 2007

Standard Borel Spaces

Definition

- *If* (X, d) *is a complete separable metric space, then the associated topological space* $(X, \mathcal{T})$ *is said to be a Polish space.*
- *A standard Borel space* $(X, \mathbf{B}(\mathcal{T}))$ *is a Polish space equipped with its* σ*-algebra* $\mathbf{B}(\mathcal{T})$ *of Borel subsets.*
- *E.g.* $\mathbb{R}$, $[0, 1]$, $\mathbb{N}^{\mathbb{N}}$, $2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$, ...

Definition

Let X, Y *be standard Borel spaces.*

- *Then the map* $\varphi : X \to Y$ *is Borel iff* $\mathrm{graph}(\varphi)$ *is a Borel subset of* $X \times Y$.
- *Equivalently,* $\varphi : X \to Y$ *is Borel iff* $\varphi^{-1}(B)$ *is a Borel set for each Borel set* $B \subseteq Y$.

Metric Spaces vs. Topological Spaces

Example

Let d_1, d_2 be the metrics on $\mathbb{R}^2$ defined by

$$d_1(\overline{x}, \overline{y}) = \sqrt{|x_1 - y_1|^2 + |x_1 - y_1|^2}$$
$$d_2(\overline{x}, \overline{y}) = |x_1 - y_1| + |x_1 - y_1|$$

Then $(\mathbb{R}^2, d_1)$, $(\mathbb{R}^2, d_2)$ induce the same topological space.

Topological Spaces vs. Standard Borel Spaces

Theorem

Let $(X, \mathcal{T})$ be a Polish space and $Y \subseteq X$ be any Borel subset. Then there exists a Polish topology $\mathcal{T}_Y \supseteq \mathcal{T}$ such that $\mathbf{B}(\mathcal{T}_Y) = \mathbf{B}(\mathcal{T})$ and Y is clopen in $(X, \mathcal{T}_Y)$.

Corollary

If $(X, \mathcal{B})$ is a standard Borel space and $Y \in \mathcal{B}$, then $(Y, \mathcal{B} \restriction Y)$ is also a standard Borel space.

Theorem (Kuratowski)

There exists a unique uncountable standard Borel space up to isomorphism.

Church's Thesis for Real Mathematics

EXPLICIT = BOREL

Borel equivalence relations

Definition

Let X be a standard Borel space. Then a *Borel equivalence relation* *on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2.*

Definition

Let G be a Polish group. Then a *standard Borel G-space* *is a standard Borel space X equipped with a Borel action $(g, x) \mapsto g \cdot x$. The corresponding G-orbit equivalence relation is denoted by E_G^X.*

Observation

If G is a countable (discrete) group and X is a standard Borel G-space, then E_G^X is a Borel equivalence relation.

The standard Borel space of countable graphs

- Let $\mathcal{C}$ be the set of graphs of the form $\Gamma = \langle \mathbb{N}, E \rangle$.
- Identify each graph $\Gamma \in \mathcal{C}$ with its edge relation $E \in 2^{\mathbb{N}^2}$.
- Then $\mathcal{C}$ is a Borel subset of $2^{\mathbb{N}^2}$ and hence is a standard Borel space.
- The isomorphism relation on $\mathcal{C}$ is the orbit equivalence relation of the natural action of $\mathrm{Sym}(\mathbb{N})$ on $\mathcal{C}$.

Remark

More generally, if σ is a sentence of $\mathcal{L}_{\omega_1,\omega}$ then

$$\mathrm{Mod}(\sigma) = \{ M = \langle \mathbb{N}, \cdots \rangle \mid M \models \sigma \}$$

is a standard Borel space.

Torsion-free abelian groups of finite rank

Definition

- *For each $n \geq 1$, let $\mathbb{Q}^n = \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{n \text{ times}}$.*
- *The standard Borel space of torsion-free abelian groups of rank n is defined to be*

$$R(\mathbb{Q}^n) = \{A \leqslant \mathbb{Q}^n \mid A \text{ contains a basis of } \mathbb{Q}^n\}.$$

Remark

If $A, B \in R(\mathbb{Q}^n)$, then

$$A \cong B \quad \text{iff} \quad \text{there exists } \varphi \in GL_n(\mathbb{Q}) \text{ such that } \varphi(A) = B.$$

The Polish space of f.g. groups

Let $\mathbb{F}_m$ be the free group on $\{x_1, \cdots, x_m\}$ and let $\mathcal{G}_m$ be the compact space of normal subgroups of $\mathbb{F}_m$. Since each m-generator group can be realised as a quotient $\mathbb{F}_m/N$ for some $N \in \mathcal{G}_m$, we can regard $\mathcal{G}_m$ as the space of m-generator groups. There are natural embeddings

$$\mathcal{G}_1 \hookrightarrow \mathcal{G}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{G}_m \hookrightarrow \cdots$$

and we can regard

$$\mathcal{G} = \bigcup_{m \geq 1} \mathcal{G}_m$$

as the space of f.g. groups.

A slight digression

Some Isolated Points

- Finite groups
- Finitely presented simple groups

The Next Stage

- $SL_3(\mathbb{Z})$

Question (Grigorchuk)

What is the Cantor-Bendixson rank of $\mathcal{G}_m$*?*

The isomorphism relation on $\mathcal{G}$

The natural action of the countable group $\mathrm{Aut}(\mathbb{F}_m)$ on $\mathbb{F}_m$ induces a corresponding homeomorphic action on the compact space $\mathcal{G}_m$ of normal subgroups of $\mathbb{F}_m$. Furthermore, each $\pi \in \mathrm{Aut}(\mathbb{F}_m)$ extends to a homeomorphism of the space $\mathcal{G}$ of f.g. groups.

If $N, M \in \mathcal{G}_m$ and there exists $\pi \in \mathrm{Aut}(\mathbb{F}_m)$ such that $\pi(N) = M$, then $\mathbb{F}_m/N \cong \mathbb{F}_m/M$. Unfortunately, the converse does not hold.

The isomorphism relation on $\mathcal{G}$

Theorem (Tietze)

If $N, M \in \mathcal{G}_m$, then the following are equivalent:

- $\mathbb{F}_m/N \cong \mathbb{F}_m/M$.
- *There exists* $\pi \in \mathrm{Aut}(\mathbb{F}_{2m})$ *such that* $\pi(N) = M$.

Corollary (Champetier)

The isomorphism relation $\cong$ on the space $\mathcal{G}$ of f.g. groups is the orbit equivalence relation arising from the homeomorphic action of the countable group $\mathrm{Aut}_f(\mathbb{F}_\infty)$ of finitary automorphisms of the free group $\mathbb{F}_\infty$ on $\{x_1, x_2, \cdots, x_m, \cdots\}$.

Borel reductions

Definition

Let E, F be Borel equivalence relations on the standard Borel spaces X, Y respectively.

- $E \leq_B F$ *iff there exists a Borel map* $f : X \to Y$ *such that*

$$x \, E \, y \iff f(x) \, F \, f(y).$$

  In this case, f is called a **Borel reduction** *from E to F.*

- $E \sim_B F$ *iff both* $E \leq_B F$ *and* $F \leq_B E$.
- $E <_B F$ *iff both* $E \leq_B F$ *and* $E \nsim_B F$.

Definition

More generally, $f : X \to Y$ *is a* **Borel homomorphism** *from E to F iff*

$$x \, E \, y \implies f(x) \, F \, f(y).$$

Smooth equivalence relations

Theorem (Silver)

If E is a Borel equivalence relation with uncountably many classes, then $\mathrm{id}_{\mathbb{R}} \leq_B E$.

Definition

The Borel equivalence relation E is smooth iff $E \leq_B \mathrm{id}_X$ for some/every uncountable standard Borel space X.

Example

The isomorphism problem on the space of countable divisible abelian groups is smooth.

E_0 is not smooth

Definition

E_0 is the Borel equivalence relation defined on $2^{\mathbb{N}}$ by:

$$x \; E_0 \; y \quad \text{iff} \quad x(n) = y(n) \text{ for all but finitely many } n.$$

- Suppose that $f : 2^{\mathbb{N}} \to [0,1]$ is a Borel reduction from E_0 to $\text{id}_{[0,1]}$.
- Let μ be the usual product probability measure on $2^{\mathbb{N}}$.
- Then $f^{-1}([0,1/2])$ and $f^{-1}([1/2,1])$ are Borel tail events.
- By Kolmogorov's zero-one law, either $\mu(f^{-1}([0,1/2])) = 1$ or $\mu(f^{-1}([1/2,1])) = 1$.
- Continuing in this fashion, f is μ-a.e. constant, which is a contradiction.

Structural vs. informational complexity

Example

Let $\equiv$ be the equivalence relation defined on the space $\mathcal{G}$ of finitely generated groups by

$$G \equiv H \quad \text{iff} \quad \mathrm{Th}\, G = \mathrm{Th}\, H.$$

Then $\equiv$ is smooth.

Observation

If E, F are Borel equivalence relations on the standard Borel spaces X, Y, then $E \leq_B F$ iff there exists a "Borel embedding" $X/E \to Y/F$.

Countable Borel equivalence relations

Definition

E be a *countable* *Borel equivalence relation iff every E-class is countable.*

Standard Example

Let G be a countable (discrete) group and let X be a standard Borel G-space. Then the corresponding orbit equivalence relation E_G^X is a countable Borel equivalence relation.

Theorem (Feldman-Moore)

If E is a countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.

Sketch Proof of Feldman-Moore

- Clearly $E \subseteq X^2$ has countable sections.
- By the Lusin-Novikov Uniformization Theorem,

$$E = \bigcup_{n \in \mathbb{N}} F_n,$$

  where each F_n is the graph of an injective partial Borel function $f_n : \text{dom}\, f_n \to X$.
- Each f_n is easily modified into a Borel bijection $g_n : X \to X$ with the same "orbits".
- Hence E is the orbit equivalence arising from the Borel action of the group

$$G = \langle g_n \mid n \in \mathbb{N} \rangle.$$

The Turing equivalence relation

Definition

The Turing equivalence relation $\equiv_T$ *on* $\mathcal{P}(\mathbb{N})$ *is defined by*

$$A \equiv_T B \quad \text{iff} \quad A \leq_T B \ \& \ B \leq_T A,$$

where $\leq_T$ *denotes Turing reducibility.*

Remark

Clearly $\equiv_T$ is a countable Borel equivalence relation on $\mathcal{P}(\mathbb{N})$.

Vague Question

Can $\equiv_T$ *be realised as the orbit equivalence relation of a "nice" Borel action of some countable group?*

Universal countable Borel equivalence relations

Definition

A countable Borel equivalence relation E is universal iff $F \leq_B E$ for every countable Borel equivalence relation F.

Theorem (Dougherty-Jackson-Kechris)

There exists a universal countable Borel equivalence relation.

Remark

On the other hand, there does not exist a universal Borel equivalence relation.

Universal countable Borel equivalence relations

Definition

- *Let $\mathbb{F}_\omega$ be the free group on infinitely many generators.*
- *Define a Borel action of $\mathbb{F}_\omega$ on*

$$(2^{\mathbb{N}})^{\mathbb{F}_\omega} = \{p \mid p : \mathbb{F}_\omega \to 2^{\mathbb{N}}\}$$

by setting

$$(g \cdot p)(h) = p(g^{-1}h), \quad p \in (2^{\mathbb{N}})^{\mathbb{F}_\omega},$$

and let E_ω be the corresponding orbit equivalence relation.

Claim

E_ω is a universal countable Borel equivalence relation.

Universal countable Borel equivalence relations

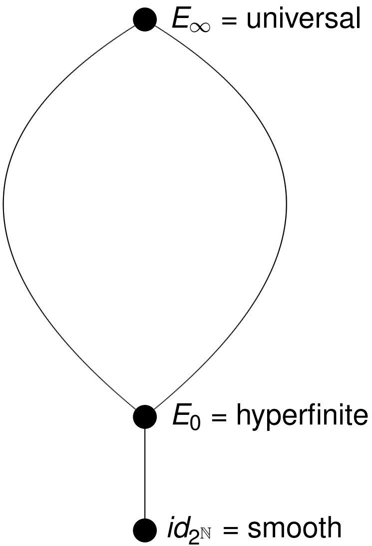
- Let E be a countable Borel equivalence relation on X.
- Then E is the orbit equivalence relation of a Borel action of $\mathbb{F}_\omega$.
- Let $\{U_i\}_{i \in \mathbb{N}}$ be a sequence of Borel subsets of X which separates points and define $f : X \to (2^{\mathbb{N}})^{\mathbb{F}_\omega}$ by $x \mapsto f_x$, where

$$f_x(h)(i) = 1 \quad \text{iff} \quad x \in h(U_i).$$

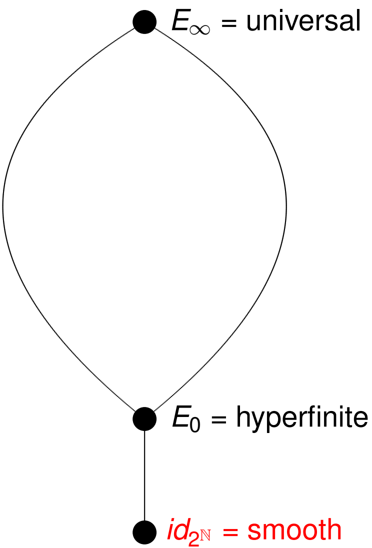
- Then f is injective and

$$\begin{aligned}
(g \cdot f_x)(h)(i) = 1 \quad & \text{iff} \quad f_x(g^{-1}h)(i) = 1 \\
& \text{iff} \quad x \in g^{-1}h(U_i) \\
& \text{iff} \quad g \cdot x \in h(U_i) \\
& \text{iff} \quad f_{g \cdot x}(h)(i) = 1.
\end{aligned}$$

Countable Borel equivalence relations



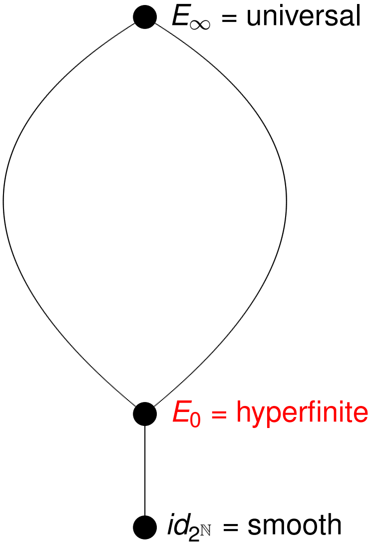
Countable Borel equivalence relations



Definition

The Borel equivalence relation E is smooth iff $E \leq_B id_{2^{\mathbb{N}}}$, where $2^{\mathbb{N}}$ is the space of infinite binary sequences.

Countable Borel equivalence relations



Definition

E_0 is the equivalence relation of eventual equality on the space $2^{\mathbb{N}}$ of infinite binary sequences.

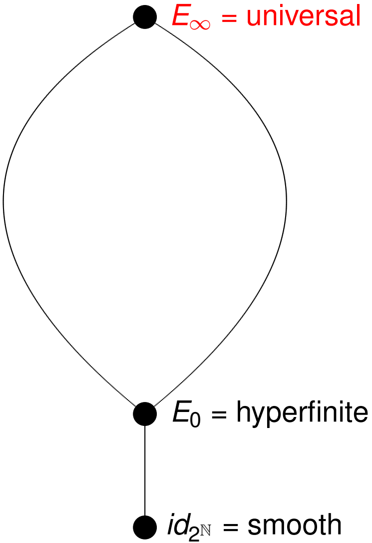
Theorem (HKL)

If E is nonsmooth Borel, then $E_0 \leq_B E$.

Theorem (DJK)

If E is countable Borel, then E can be realized by a Borel $\mathbb{Z}$-action iff $E \leq_B E_0$.

Countable Borel equivalence relations



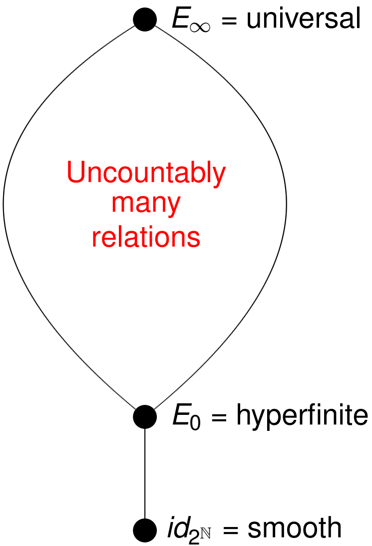
Definition

A countable Borel equivalence relation E is universal iff $F \leq_B E$ for every countable Borel equivalence relation F.

Theorem (JKL)

The orbit equivalence relation E_∞ of the action of the free group $\mathbb{F}_2$ on its powerset $\mathcal{P}(\mathbb{F}_2)$ is countable universal.

Countable Borel equivalence relations



Theorem (Adams-Kechris 2000)

There exist $2^{\aleph_0}$ many countable Borel equivalence relations up to Borel bireducibility.

Question

Where does $\equiv_T$ fit into this picture?

The Kechris Conjecture

$\equiv_T$ is universal.

The Martin Conjecture

$\equiv_T$ is not universal.

Martin's Theorem

Definition

The set of Turing degrees is defined to be

$$\mathcal{D} = \{\mathbf{a} = [A]_{\equiv_T} \mid A \in \mathcal{P}(\mathbb{N})\}.$$

Definition

A subset $X \subseteq \mathcal{D}$ is said to be Borel iff

$$X^* = \bigcup \{\mathbf{a} \mid \mathbf{a} \in X\}$$

is a Borel subset of $\mathcal{P}(\mathbb{N})$.

Remark

$\mathcal{D}$ is not a standard Borel space.

Martin's Theorem

Example

For each $\mathbf{a} \in \mathcal{D}$, the corresponding cone $C_{\mathbf{a}} = \{\mathbf{b} \in \mathcal{D} \mid \mathbf{a} \leq \mathbf{b}\}$ is a Borel subset of $\mathcal{D}$.

Definition

If $\mathbf{a}, \mathbf{b} \in \mathcal{D}$, then $\mathbf{a} \leq \mathbf{b}$ iff $A \leq_T B$ for each $A \in \mathbf{a}$, $B \in \mathbf{b}$.

Theorem (Martin)

If $X \subseteq \mathcal{D}$ is Borel, then for some $\mathbf{a} \in \mathcal{D}$, either $C_{\mathbf{a}} \subseteq X$ or $C_{\mathbf{a}} \subseteq \mathcal{D} \smallsetminus X$.

Proof of Martin's Theorem

- Let $X \subseteq \mathcal{D}$ be Borel and consider the game $G(X^*)$

$$a = a(0)\, a(1)\, a(2) \cdots \quad \text{where each } a(n) \in 2$$

  such that Player I wins iff $a \in X^*$.
- $G(X^*)$ is Borel and hence is determined.
- Suppose that $\varphi : 2^{<\mathbb{N}} \to 2$ is a winning strategy for Player I.
- We claim that $C_\varphi \subseteq X$.
- Suppose $\varphi \leq_T x$ and let Player II play $x = a(1)\, a(3)\, a(5) \cdots$
- Then $y = \varphi(x) \in X^*$ and $x \equiv_T y$. Hence $x \in X^*$.

Remark

For later use, note that $C_{\mathbf{a}} \subseteq X$ iff X is cofinal in $\mathcal{D}$.

The Martin Conjecture

Definition

A function $f : \mathcal{D} \to \mathcal{D}$ *is Borel iff there exists a Borel function* $\varphi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ *such that* $f([A]_{\equiv_T}) = [\varphi(A)]_{\equiv_T}$.

Example

The jump operator $\mathbf{a} \mapsto \mathbf{a}'$ is a Borel function on $\mathcal{D}$.

The Martin Conjecture

If $f : \mathcal{D} \to \mathcal{D}$ *is Borel, then either* f *is constant on a cone or else* $f(\mathbf{a}) \geq \mathbf{a}$ *on a cone.*

Some partial results

Theorem (Slaman-Steel)

If $f : \mathcal{D} \to \mathcal{D}$ is Borel and $f(\mathbf{a}) < \mathbf{a}$ on a cone, then f is constant on a cone.

Theorem (Slaman-Steel)

If the Borel map $f : \mathcal{D} \to \mathcal{D}$ is uniformly invariant, then either f is constant on a cone or else $f(\mathbf{a}) \geq \mathbf{a}$ on a cone.

Slightly Inaccurate Definition

A Borel function is uniformly invariant iff there exists a function $t : \omega \times \omega \to \omega \times \omega$ such that "on a cone"

$$A = \{i\}^B, B = \{j\}^A \implies f(A) = \{t_1(i,j)\}^{f(B)}, f(B) = \{t_2(i,j)\}^{f(A)}.$$

Kechris vs. Martin

- If $\equiv_T$ is universal, then $(\equiv_T \times \equiv_T) \sim_B \equiv_T$.
- Hence there exist Borel complete sections $Y \subseteq \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$, $Z \subseteq \mathcal{P}(\mathbb{N})$ and a Borel isomorphism

$$f : \langle\, Y, (\equiv_T \times \equiv_T) \restriction Y \,\rangle \to \langle\, Z, \equiv_T \restriction Z \,\rangle.$$

- This induces a Borel pairing function $f : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$.
- Fix $\mathbf{d}_0 \neq \mathbf{d}_1 \in \mathcal{D}$ and define the Borel maps $f_i : \mathcal{D} \to \mathcal{D}$ by $f_i(\mathbf{a}) = f(\mathbf{d}_i, \mathbf{a})$.
- By the Martin Conjecture, $f_i(\mathbf{a}) \geq \mathbf{a}$ on a cone and so ran f_i are cofinal Borel subsets of $\mathcal{D}$.
- Hence each ran f_i contains a cone, which is impossible since ran $f_0 \cap$ ran $f_1 = \emptyset$.

The Arithmetic equivalence relation

Definition

The arithmetic equivalence relation $\equiv_A$ *on* $\mathcal{P}(\mathbb{N})$ *is defined by*

$$B \equiv_A C \quad \text{iff} \quad B \leq_A C \;\&\; C \leq_A B,$$

where $\leq_A$ *denotes arithmetic reducibility.*

Theorem (Slaman-Steel)

$\equiv_A$ *is a universal countable Borel equivalence relation.*

Remark (Slaman)

The difference between the two cases is that the arithmetic degrees have less closure with respect to arithmetic equivalences than the Turing degrees do for recursive equivalences.