

# THE HOD DICHOTOMY

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## 1. INTRODUCTION

This paper provides a more accessible account of some of the material from Woodin [4] and [5]. All unattributed results are due to the first author.

Recall that  $0^\#$  is a certain set of natural numbers that codes an elementary embedding  $j : L \rightarrow L$  such that  $j \neq \text{id} \upharpoonright L$ . Jensen's covering lemma says that if  $0^\#$  does not exist and  $A$  is an uncountable set of ordinals, then there exists  $B \in L$  such that  $A \subseteq B$  and  $|A| = |B|$ . The conclusion implies that if  $\gamma$  is a singular cardinal, then it is a singular cardinal in  $L$ . It also implies that if  $\gamma \geq \omega_2$  and  $\gamma$  is a successor cardinal in  $L$ , then  $\text{cf}(\gamma) = |\gamma|$ . In particular, if  $\beta$  is a singular cardinal, then  $(\beta^+)^L = \beta^+$ . Intuitively, this says that  $L$  is close to  $V$ . On the other hand, should  $0^\#$  exist, if  $\gamma$  is an uncountable cardinal, then  $\gamma$  is an inaccessible cardinal in  $L$ . In this case, we could say that  $L$  is far from  $V$ . Thus, the covering lemma has the following corollary, which does not mention  $0^\#$ .

**Theorem 1** (Jensen). *Exactly one of the following holds.*

- (1)  *$L$  is correct about singular cardinals and computes their successors correctly.*
- (2) *Every uncountable cardinal is inaccessible in  $L$ .*

Imagine an alternative history in which this  $L$  dichotomy was discovered without knowledge of  $0^\#$  or more powerful large cardinals. Clearly, (1) is consistent because it holds in  $L$ . On the other hand, whether or not there is a proper class of inaccessible cardinals in  $L$  is absolute to generic extensions. This incomplete evidence might have led set theorists to conjecture that (2) fails. Of course, (2) only holds when  $0^\#$  exists but  $0^\#$  does not belong to  $L$  and  $0^\#$  cannot be added by forcing.

Canonical inner models other than  $L$  have been defined and shown to satisfy similar covering properties and corresponding dichotomies. Part of what makes them canonical is that they are contained in HOD. In these notes, we will prove a dichotomy theorem of this kind for HOD itself. Towards the formal statement, recall that a cardinal  $\delta$  is *extendible* iff for every  $\eta > \delta$ , there exists  $\theta > \eta$  and an elementary embedding  $j : V_{\eta+1} \rightarrow V_{\theta+1}$  such that  $\text{crit}(j) = \delta$  and  $j(\delta) > \eta$ . The following result expresses the idea that either HOD is close to  $V$  or else HOD is far from  $V$ . We will refer to it as the HOD Dichotomy.

**Theorem 2.** *Assume that  $\delta$  is an extendible cardinal. Then exactly one of the following holds.*

- (1) *For every singular cardinal  $\gamma > \delta$ ,  $\gamma$  is singular in HOD and  $(\gamma^+)^{\text{HOD}} = \gamma^+$ .*
- (2) *Every regular cardinal greater than  $\delta$  is measurable in HOD.*

In this note, we shall prove a dichotomy in which (2) is weakened to hold for all sufficiently large regular cardinals greater than  $\delta$ ; see Corollary 20. The full result can be found in [4] Theorem 212.

Notice that we have stated the HOD dichotomy without deriving it from a covering property that involves a “large cardinal missing from HOD”. In other words, no analogue of  $0^\#$  is mentioned and the alternative history we described for  $L$  is what has actually happened in the case of HOD. This leads us to conjecture that (2) fails. One reason is that (2) is absolute between  $V$  and its generic extensions by posets that belong to  $V_\delta$ , which we will show this in the next section. There is some evidence for this conjecture. All known large cardinal axioms (which do not contradict the Axiom of Choice) are compatible with  $V = \text{HOD}$  and so trivially cannot imply (2). Further, we shall see that the main technique for obtaining independence in set theory (forcing) probably cannot be used to show that (2) is relatively consistent with the existence of an extendible cardinal starting from any known large cardinal hypothesis which is also consistent with the Axiom of Choice. Finally, by definition HOD contains all definable sets of ordinals and this makes it difficult to imagine a meaningful analogue of  $0^\#$  for HOD.

Besides evidence in favor of this conjecture about HOD, we also have applications. Recall that Kunen proved in ZFC that there is no non-trivial elementary embedding from  $V$  to itself. It is a longstanding open question whether this is a theorem of ZF alone. One of our applications is progress on this problem. This and other applications will be listed in Section 7.

## 2. GENERIC ABSOLUTENESS

In this section, we establish some basic properties of forcing and HOD, and use them to show that the conjecture about HOD from the previous section is absolute to generic extensions. In other words, if  $\mathbb{P}$  is a poset, then clause (2) of Theorem 2 holds in  $V$  iff it holds in every generic extension by  $\mathbb{P}$ .

First observe that if  $\mathbb{P}$  is a weakly homogeneous (see [1] Theorem 26.12) and ordinal definable poset in  $V$ , and  $G$  is a  $V$ -generic filter on  $\mathbb{P}$ , then  $\text{HOD}^{V[G]} \subseteq \text{HOD}^V$ . This is immediate from the basic fact about weakly homogeneous forcing that for all  $x_1, \dots, x_n \in V$  and formula  $\varphi(v_1, \dots, v_n)$ , every condition in  $\mathbb{P}$  decides  $\varphi(\check{x}_1, \dots, \check{x}_n)$  the same way. We also use here that a class model of ZFC can be identified solely from its sets of ordinals, since each level of its  $V$  hierarchy can, using the Axiom of Choice, be encoded by a relation on  $|V_\alpha|$  and then recovered by collapsing. We shall use this fact repeatedly.

Let us pause to give an example of the phenomenon we just mentioned in which HOD of the generic extension is properly contained in HOD of the ground model. Let  $\mathbb{P}$  be Cohen forcing and  $g : \omega \rightarrow \omega$  be a Cohen real over  $L$ . Of course,  $g \notin L$ . In  $L[g]$ , let  $\mathbb{Q}$  be the Easton poset that forces

$$2^{\omega_n} = \begin{cases} \omega_{n+1} & g(n) = 0 \\ \omega_{n+2} & g(n) = 1. \end{cases}$$

Both  $\mathbb{P}$  and  $\mathbb{Q}$  are cardinal preserving. Now let  $H$  be an  $L[g]$ -generic filter on  $\mathbb{Q}$ . Observe that  $g \in \text{HOD}^{L[g][H]}$  because it can be read off from  $\kappa \mapsto 2^\kappa$  in  $L[g][H]$ . Now let  $\lambda$  be a regular cardinal greater than  $|\mathbb{P} * \mathbb{Q}|$ . Then  $\mathbb{P} * \mathbb{Q} * \text{Coll}(\omega, \lambda)$  and  $\text{Coll}(\omega, \lambda)$  have isomorphic Boolean completions, so there is an  $L$ -generic filter  $J$  on  $\text{Coll}(\omega, \lambda)$  and an  $L[g][H]$ -generic filter  $I$  on  $\text{Coll}(\omega, \lambda)$  such that  $L[J] = L[g][H][I]$ . Using the fact that  $\text{Coll}(\omega, \lambda)$  is definable and weakly homogeneous we see that

$$L = \text{HOD}^{L[J]} = \text{HOD}^{L[g][H][I]} \subsetneq \text{HOD}^{L[g][H]}$$

where the inequality is witnessed by the Cohen real  $g$ .

An important fact about forcing which was discovered relatively recently is that if  $\delta$  is a regular uncountable cardinal and  $\mathbb{P} \in V_\delta$  is a poset, then  $V$  is definable from  $\mathcal{P}(\delta) \cap V$  in  $V[G]$ . Towards the precise statement and proof, we make the following definitions.

**Definition 3.** Let  $\delta$  be a regular uncountable cardinal and  $N$  be a transitive class model of ZFC. Then

- $N$  has the  $\delta$ -covering property iff for every  $\sigma \subseteq N$  with  $|\sigma| < \delta$ , there exists  $\tau \in N$  such that  $|\tau| < \delta$  and  $\tau \supseteq \sigma$ , and
- $N$  has the  $\delta$ -approximation property iff for every cardinal  $\kappa$  with  $\text{cf}(\kappa) \geq \delta$  and every  $\subseteq$ -increasing sequence of sets  $\langle \tau_\alpha \mid \alpha < \kappa \rangle$  from  $N$ ,  $\bigcup \tau_\alpha \in N$ .

By Jensen's theorem,  $L$  has the  $\delta$ -covering property in  $V$  for every regular  $\delta > \omega$  if  $0^\#$  does not exist. Next, we show that  $V$  has covering and approximation properties in its generic extensions.

**Lemma 4.** Let  $\delta > \omega$  regular and  $\mathbb{P}$  a poset with  $|\mathbb{P}| < \delta$ . Then  $V$  has  $\delta$ -covering and  $\delta$ -approximation in  $V[G]$  whenever  $G$  is a  $V$ -generic filter on  $\mathbb{P}$ .

*Proof.* First, we show the covering property. Let  $\sigma$  be a name such that  $\Vdash \sigma \subset V$  and  $|\sigma| < \delta$ . By the  $\delta$  chain condition, there are fewer than  $\delta$  possible values of  $|\sigma|$ . Let  $\gamma < \delta$  be the supremum of these and pick  $\dot{f}$  such that  $\Vdash \dot{f} : \gamma \rightarrow \sigma$ . To finish this part of the proof, let  $\tau$  be the set of possible values for  $\dot{f}(\alpha)$  and  $\alpha < \gamma$ .

Second, we prove the approximation property. Say  $p$  forces that  $\text{cf}(\kappa) \geq \delta$  and  $\langle \tau_\alpha \mid \alpha < \kappa \rangle$  is an increasing sequence of sets from  $V$ . For  $\alpha < \kappa$ , let  $p_\alpha$  decide the value of  $\tau_\alpha$ . Because  $|\mathbb{P}| < \delta \leq \text{cf}(\kappa) \leq \kappa$  there must be some  $p_\beta$

that is repeated cofinally often and so determines  $\bigcup \tau_\alpha$ , thereby forcing the union to belong to  $V$ . By density, the union is forced to belong to  $V$ .  $\square$

The next theorem is the promised result on the definability of the ground model, which we state somewhat more generally. Part (1) is due to Hamkins and (2) to Laver and Woodin independently.

**Theorem 5.** *Let  $\delta$  be a regular uncountable cardinal. Suppose that  $M$  and  $N$  are transitive class models of ZFC that satisfy the  $\delta$ -covering and  $\delta$ -approximation properties,  $\delta^+ = (\delta^+)^N = (\delta^+)^M$ , and*

$$N \cap \mathcal{P}(\delta) = M \cap \mathcal{P}(\delta).$$

(1) *Then  $M = N$ .*

(2) *In particular,  $N$  is  $\Sigma_2$ -definable from  $N \cap \mathcal{P}(\delta)$ .*

*Proof.* For part (1) we show by recursion on ordinals  $\gamma$  that for all  $A \subseteq \gamma$

$$A \in M \iff A \in N.$$

The case  $\gamma \leq \delta$  is clear. By the induction hypothesis,  $M$  and  $N$  have the same cardinals  $\leq \gamma$ , and, if  $\gamma$  is not a cardinal in these models, then they have the same power set of  $\gamma$ . Thus, we may assume that  $\gamma$  is a cardinal of both  $M$  and  $N$ .

**Case 1.**  $\text{cf}(\gamma) \geq \delta$

Then,  $A \in M$  iff  $A \cap \alpha \in M$  for every  $\alpha < \gamma$ . The forward direction is clear. For the reverse, use the  $\delta$ -approximation property to see

$$A = \bigcup \{A \cap \alpha \mid \alpha < \gamma\} \in M.$$

The same holds for  $N$ .

**Case 2.**  $\gamma > \delta$ ,  $\text{cf}(\gamma) < \delta$  and  $|A| < \delta$

Define increasing sequences  $\langle E_\alpha \mid \alpha < \delta \rangle$  and  $\langle F_\alpha \mid \alpha < \delta \rangle$  of subsets of  $\gamma$  such that  $|E_\alpha|, |F_\alpha| < \delta$ ,  $A \subseteq E_0$ ,  $E_\alpha \subseteq F_\alpha$ ,  $\bigcup_{\alpha < \beta} F_\alpha \subseteq E_\beta$ ,  $E_\alpha \in M$  and  $F_\alpha \in N$ . For the construction, use the  $\delta$ -covering property alternately for  $M$  and  $N$ . Then define  $E = \bigcup E_\alpha = \bigcup F_\alpha$  and note that  $E \in M \cap N$  by  $\delta$ -approximation property. Let  $\theta$  be the order-type of  $E$  and  $\pi : E \rightarrow \theta$  the Mostowski collapse. Then  $\pi \in M \cap N$ . Also,  $\theta < \delta^+ = (\delta^+)^M = (\delta^+)^N$  because  $|E| \leq \delta$ . By the induction hypothesis,

$$A \in M \iff \pi[A] \in M \iff \pi[A] \in N \iff A \in N.$$

**Case 3.**  $\gamma > \delta$ ,  $\text{cf}(\gamma) < \delta$  and  $|A| \geq \delta$

We claim that  $A \in M$  iff

- (i)<sub>M</sub> for every  $\alpha < \gamma$ ,  $A \cap \alpha \in M$  and
- (ii)<sub>M</sub> for every  $\sigma \subseteq \gamma$ , if  $|\sigma| < \delta$  and  $\sigma \in M$ , then  $A \cap \sigma \in M$ .

We also claim that  $A \in N$  iff (i)<sub>N</sub> and (ii)<sub>N</sub>. The induction hypothesis is that (i)<sub>M</sub> iff (i)<sub>N</sub> and in case (2) we showed that (ii)<sub>M</sub> iff (ii)<sub>N</sub>, so our claim implies  $A \in M$  iff  $A \in N$  as desired.

The forward implication of the claim is obvious, so assume (i)<sub>M</sub> and (ii)<sub>M</sub>. Pick  $\theta$  with  $\text{cf}(\theta) > \gamma$  and the defining formula for  $M$  absolute to  $V_\theta$ . Define an increasing chain  $\langle X_\alpha \mid \alpha < \delta \rangle$  of elementary substructures of  $V_\theta$  and an increasing chain  $\langle Y_\alpha \mid \alpha < \delta \rangle$  of subsets of  $V_\theta \cap M$  such that  $|X_\alpha|, |Y_\alpha| < \delta$ ,  $A \in X_0$ ,  $\text{sup}(X_0 \cap \gamma) = \gamma$ ,  $X_\alpha \cap N \subseteq Y_\alpha$ ,  $Y_\alpha \in M$  and  $\bigcup_{\alpha < \beta} (Y_\alpha \cup X_\alpha) \subseteq X_\beta$ . We use Downward Lowenheim-Skolem to obtain  $X_\alpha$  and the  $\delta$ -covering property to obtain  $Y_\alpha$ . Define  $X = \bigcup X_\alpha$  and  $Y = \bigcup Y_\alpha$ . Then  $X \prec V_\theta$  and  $Y = X \cap M \prec V_\theta \cap M$ . By the  $\delta$ -approximation property,  $Y \in M$ . By (ii)<sub>M</sub>, for every  $\alpha < \delta$ ,  $A \cap Y_\alpha \in M$ . Again, by the  $\delta$ -approximation property,  $A \cap Y \in M$ . Now consider an arbitrary  $\alpha \in Y \cap \gamma$  and observe that

- $A \cap \alpha \in Y$  because  $A \in X$  so  $A \cap \alpha \in X$ , and  $A \cap \alpha \in M$  by (ii)<sub>M</sub>; and
- for every  $b \in Y$ , if  $b \cap Y = (A \cap Y) \cap \alpha$ , then  $Y \models b = A \cap \alpha$ , so  $b = A \cap \alpha$ .

Here we have used (i)<sub>M</sub> and  $Y \prec V_\theta \cap M$ . So the sequence  $\langle A \cap \alpha \mid \alpha \in Y \cap \gamma \rangle$  is definable in  $M$  from parameters  $\gamma$ ,  $Y$  and  $A \cap Y$ . In particular, this function belongs to  $M$ . The union of its range is  $A$ , so  $A \in M$ .

Part (2) now follows.  $A \in N$  iff there is a large regular  $\theta$  and a model  $M \subseteq V_\theta$  of ZFC-Power Set satisfying  $\delta$ -covering and  $\delta$ -approximation in  $V_\theta$  such that  $M \cap \mathcal{P}(\delta) = N \cap \mathcal{P}(\delta)$  and  $A \in M$ . This is a  $\Sigma_2$  statement.  $\square$

We will use the following amazing result. The final equality is not as well known, so we include a proof. Note that  $\text{OD}_{\mathcal{A}}$  here denotes the class of all sets that are definable using ordinals and members of  $\mathcal{A}$ , and  $\text{HOD}_{\mathcal{A}}$  is defined correspondingly.

**Theorem 6** (Vopěnka). *For every ordinal  $\kappa$ , there exists  $\mathbb{B} \in \text{HOD}$  such that*

$$\text{HOD} \models \mathbb{B} \text{ is a complete Boolean algebra}$$

*and, for every  $a \subseteq \kappa$ , there exists a HOD-generic filter  $G$  on  $\mathbb{B}$  such that*

$$\text{HOD}[a] \subseteq \text{HOD}_{\{G\}} = \text{HOD}_{\{a\}} = \text{HOD}[G].$$

*Proof.* First define  $\mathbb{B}^*$  to be  $\mathcal{P}(\mathcal{P}(\kappa)) \cap \text{OD}$  with its Boolean algebra structure. Then  $\mathbb{B}^* \in \text{OD}$  and  $\mathbb{B}^*$  is OD-complete. Given  $a \subseteq \kappa$ , we let

$$G^* = \{X \in \mathbb{B}^* \mid a \in X\}$$

and see that  $G^*$  is an OD-generic filter on  $\mathbb{B}^*$ . Fix a definable bijection  $\pi$  from  $\mathcal{P}(\mathcal{P}(\kappa)) \cap \text{OD}$  to an ordinal. Define  $\mathbb{B}$  so that  $\pi : \mathbb{B}^* \simeq \mathbb{B}$ . Let  $G = \pi[G^*]$ . Then  $G$  is a HOD-generic filter on  $\mathbb{B}$ . It is straightforward to see that  $G \in \text{HOD}_{\{a\}}$  so it remains to see that  $\text{HOD}_{\{a\}} \subseteq \text{HOD}[G]$ . Let

$S \in \text{HOD}_{\{a\}}$ ; we may assume  $S$  is a set of ordinals. Say

$$S = \{\zeta < \theta \mid V_\theta \models \varphi(\zeta, \eta_1, \dots, \eta_n, a)\}.$$

For each  $\zeta < \theta$ , let

$$X_\zeta = \{b \subseteq \kappa \mid V_\theta \models \varphi(\zeta, \eta_1, \dots, \eta_n, b)\}.$$

Then  $\zeta \mapsto \pi(X_\zeta)$  belongs to  $\text{HOD}$ . So  $S = \{\zeta \mid \pi(X_\zeta) \in G\}$  belongs to  $\text{HOD}[G]$ .  $\square$

Combining the results in this section, we obtain the following.

**Corollary 7.** *Let  $\mathbb{P} \in \text{OD}$  be a weakly homogeneous poset. Suppose  $G$  is a  $V$ -generic filter on  $\mathbb{P}$ . Then  $\text{HOD}^V$  is a generic extension of  $\text{HOD}^{V[G]}$ .*

*Proof.* Fix  $\delta > |\mathbb{P}|$ . By Lemma 4 and Theorem 5,  $V$  is definable in  $V[G]$  from  $A = \mathcal{P}(\delta) \cap V$ . In  $V$ , let  $\kappa = |A|$  and  $E$  be a binary relation on  $\kappa$  such that the Mostowski collapse of  $(\kappa, E)$  is  $(\text{trcl}(\{A\}), \in)$ . Then  $V_\gamma \in \text{OD}_{\{E\}}^{V[G]}$  for every  $\gamma$ , therefore

$$\text{HOD}^V \subseteq \text{HOD}_{\{E\}}^{V[G]}.$$

By Theorem 6, we have a  $\text{HOD}^{V[G]}$ -generic filter  $H$  on a Vopěnka algebra so that

$$\text{HOD}_{\{E\}}^{V[G]} = \text{HOD}^{V[G]}[H].$$

Combining all of the above gives

$$\text{HOD}^{V[G]} \subseteq \text{HOD}^V \subseteq \text{HOD}_{\{E\}}^{V[G]} = \text{HOD}^{V[G]}[H].$$

As  $\text{HOD}^V$  is nested between  $\text{HOD}^{V[G]}$  and a generic extension thereof, it is itself a generic extension of  $\text{HOD}^{V[G]}$  (see [1] Theorem 15.43).  $\square$

Finally, we discuss again our conjecture that clause (2) of Theorem 2 fails. Let us temporarily call this the HOD conjecture although a slightly different statement will get this name later. We wish to see that this conjecture is absolute between  $V$  and its generic extensions. Of course, Theorem 2 has an extendible cardinal  $\delta$  in its hypothesis. We should assume that we are forcing with a poset  $\mathbb{P} \in V_\delta$  to assure that if  $G$  is a  $V$ -generic filter on  $\mathbb{P}$ , then  $\delta$  remains extendible in  $V[G]$ .

To see this, given  $\eta > \delta$  limit observe that for each member of  $V_\eta^{V[G]}$  we can, by induction on  $\eta$  build a name in  $V_\eta$  for that member. This is done as usual for nice names by considering maximal antichains, taking advantage of the fact  $\mathbb{P}$  is small with respect to  $\eta$ . Thus  $V_\eta[G] = (V_\eta)^{V[G]}$ . Now take  $j : V_\eta \rightarrow V_\theta$  elementary and define  $\tilde{j} : V_\eta^{V[G]} \rightarrow V_\theta^{V[G]}$  by  $\tilde{j}(\tau_G) = j(\tau)_G$ . This is a variation on the proof that measurability is preserved by small forcing; see [1] Theorem 21.2 or [3] Theorem 3.

**Corollary 8.** *The following statement is absolute between  $V$  and its generic extensions by posets in  $V_\delta$ : “ $\delta$  is an extendible cardinal and for every singular cardinal  $\gamma > \delta$ ,  $\gamma$  is singular in  $\text{HOD}$  and  $(\gamma^+)^{\text{HOD}} = \gamma^+$ .”*

*Proof.* If  $\mathbb{P}$  is ordinal definable and weakly homogeneous, then it is clear from Corollary 7 that the HOD conjecture is absolute between  $V$  and  $V[G]$ . Now consider the general case. Take  $\kappa < \delta$  an inaccessible cardinal such that  $\mathbb{P} \in V_\kappa$ . Let  $J$  be a  $V[G]$ -generic filter on  $\text{Coll}(\omega, \kappa)$  and  $I$  be a  $V$ -generic filter on  $\text{Coll}(\omega, \kappa)$  such that  $V[G][I] = V[J]$ . Now  $\text{Coll}(\omega, \kappa)$  is ordinal definable and weakly homogeneous so the HOD conjecture is absolute between  $V$  and  $V[J]$ , as well as between  $V[G]$  and  $V[G][I]$ . Therefore, it is absolute between  $V$  and  $V[G]$ .  $\square$

### 3. THE HOD CONJECTURE

The official HOD Conjecture is closely related to the conjecture we have been contemplating for two sections. Intuitively, it also says that HOD is not far from  $V$ , which will turn out to mean that they are close. The HOD Conjecture involves a new concept, which we define first.

**Definition 9.** Let  $\lambda$  be an uncountable regular cardinal. Then  $\lambda$  is  $\omega$ -strongly measurable in HOD iff there is  $\kappa < \lambda$  such that

- (1)  $(2^\kappa)^{\text{HOD}} < \lambda$  and
- (2) there is no partition  $\langle S_\alpha \mid \alpha < \kappa \rangle$  of  $\text{cof}(\omega) \cap \lambda$  into stationary sets such that  $\langle S_\alpha \mid \alpha < \kappa \rangle \in \text{HOD}$ .

**Lemma 10.** Assume  $\lambda$  be  $\omega$ -strongly measurable in HOD. Then

$$\text{HOD} \models \lambda \text{ is a measurable cardinal.}$$

*Proof.* We claim that there exists a stationary set  $S \subseteq \text{cof}(\omega) \cap \lambda$  such that  $S \in \text{HOD}$  and there is no partition of  $S$  into two stationary sets that belong to HOD.

First, let us see how to finish proving the lemma based on the claim. Let  $\mathcal{F}$  be the club filter restricted to  $S$ . That is,

$$\mathcal{F} = \{X \subseteq S \mid \text{there is a club } C \text{ such that } X \supseteq C \cap S\}.$$

Let  $\mathcal{G} = \mathcal{F} \cap \text{HOD}$ . Clearly,  $\mathcal{G} \in \text{HOD}$  and

$$\text{HOD} \models \mathcal{G} \text{ is a } \lambda\text{-complete filter on } \mathcal{P}(S).$$

By the claim,

$$\text{HOD} \models \mathcal{G} \text{ is an ultrafilter on } \mathcal{P}(S).$$

Now we prove the claim by contradiction. Fix a cardinal  $\kappa < \lambda$  such that  $(2^\kappa)^{\text{HOD}} < \lambda$  and there is no partition  $\langle S_\alpha \mid \alpha < \kappa \rangle$  of  $\text{cof}(\omega) \cap \lambda$  into stationary sets such that  $\langle S_\alpha \mid \alpha < \kappa \rangle \in \text{HOD}$ . This allows us to define a subtree  $T$  of  ${}^{<\kappa}2$  with height  $\kappa + 1$  and a sequence  $\langle S_r \mid r \in T \rangle$  that belongs to HOD such that

- (1)  $S_{\langle \rangle} = \text{cof}(\omega) \cap \lambda$ ,
- (2) For every  $r \in T$ ,
  - (a)  $S_r$  is stationary,
  - (b)  $r \frown \langle 0 \rangle$  and  $r \frown \langle 1 \rangle$  belong to  $T$ ,
  - (c)  $S_r$  is the disjoint union of  $S_{r \frown \langle 0 \rangle}$  and  $S_{r \frown \langle 1 \rangle}$ , and

- (d) if  $\text{dom}(r)$  is a limit ordinal, then  $S_r = \bigcap \{S_{r \upharpoonright \alpha} \mid \alpha \in \text{dom}(r)\}$ .
- (3) For every limit ordinal  $\beta \leq \kappa$  and  $r \in {}^\beta 2 - T$ , if  $r \upharpoonright \alpha \in T$  for every  $\alpha < \beta$ , then  $\bigcap_{\alpha < \beta} S_{r \upharpoonright \alpha}$  is non-stationary.

First notice that  $\text{cof}(\omega) \cap \lambda$  belongs to HOD even though it might mean something else there. Also,  $\{S \subseteq \lambda \mid S \in \text{HOD} \text{ and } S \text{ is stationary}\}$  belongs to HOD even though there might be sets which are stationary in HOD but not actually stationary. In any case, HOD can recognise when a given  $S \in \text{HOD}$  is stationary in  $V$  and, by the putative failure of the claim, choose a partition of  $S$  into two sets which are again stationary in  $V$ . This choice is done in a uniform way using a wellordering of

$$\{S \subseteq \lambda \mid S \in \text{HOD} \text{ and } S \text{ is stationary}\}$$

in HOD. This gets us through successor stages of the construction. Suppose that  $\beta \leq \kappa$  is a limit ordinal and that we have already constructed in HOD  $\langle S_r \mid r \in T \cap {}^{<\beta} 2 \rangle$ . By (3) we have recursively maintained that, except for a non-stationary set,  $\text{cof}(\omega) \cap \lambda$  equals

$$\bigcup \left\{ \bigcap \{S_{r \upharpoonright \alpha} \mid \alpha < \beta\} \mid r \text{ is a } \beta\text{-branch of } T \cap {}^{<\beta} 2 \text{ and } r \in \text{HOD} \right\}.$$

Since the club filter over  $\lambda$  is  $\lambda$ -complete and  $|{}^\beta 2|^{\text{HOD}} < \lambda$ , there exists at least one such  $r$  for which the corresponding intersection is stationary. We put  $r \in T \cap {}^\beta 2$  and define  $S_r = \bigcap \{S_{r \upharpoonright \alpha} \mid \alpha < \beta\}$  in this case. That completes the construction. Now take any  $r \in T$  with  $\text{dom}(r) = \kappa$ . Then  $S_r$  is the disjoint union of the stationary sets  $S_{r \upharpoonright (\alpha+1)} - S_{r \upharpoonright \alpha}$  for  $\alpha < \kappa$ . This readily contradicts our choice of  $\kappa$ .  $\square$

**Definition 11.** The HOD *Conjecture* is the statement:

There is a proper class of regular cardinals that are not  $\omega$ -strongly measurable in HOD.

It turns out that if  $\delta$  is an extendible cardinal, then the HOD Conjecture is equivalent to the failure of clause (2) of the dichotomy, Theorem 2, which is the conjecture we discussed in the previous two sections. In particular, a model in which the HOD conjecture fails cannot be obtained by forcing. It is clear that if HOD is correct about singular cardinals and computes their successors correctly (clause (1) of Theorem 2) then the HOD Conjecture holds, as

$$\{\gamma^+ \mid \gamma \in \text{On} \text{ and } \gamma \text{ is a singular cardinal}\}$$

is a proper class of regular cardinals which are not  $\omega$ -strongly measurable in HOD.

We close this section with additional remarks on the status of the HOD Conjecture.

- (i) It is not known whether more than 3 regular cardinals which are  $\omega$ -strongly measurable in HOD can exist.
- (ii) Suppose  $\gamma$  is a singular cardinal,  $\text{cof}(\gamma) > \omega$  and  $|V_\gamma| = \gamma$ . It is not known whether  $\gamma^+$  can be  $\omega$ -strongly measurable in HOD.



- (iii) Let  $\delta$  be a supercompact cardinal. It is not known whether any regular cardinal above  $\delta$  can be  $\omega$ -strongly measurable in HOD.

#### 4. SUPERCOMPACTNESS

Recall that a cardinal  $\delta$  is  $\gamma$ -supercompact iff there is a transitive class  $M$  and an elementary embedding  $j : V \rightarrow M$  such that  $\text{crit}(j) = \delta$ ,  $j(\delta) > \gamma$  and  ${}^\gamma M \subseteq M$ . Also,  $\delta$  is a supercompact cardinal iff  $\delta$  is  $\gamma$ -supercompact cardinal for every  $\gamma > \delta$ . If  $\delta$  is an extendible cardinal, then  $\delta$  is supercompact and  $\{\alpha < \delta \mid \alpha \text{ is supercompact}\}$  is stationary in  $\delta$  (see [2] Theorem 23.7).

There is a standard first-order way to express supercompactness in terms of measures, which we review. First suppose that  $j : V \rightarrow M$  witnesses that  $\delta$  is a  $\gamma$ -supercompact. Observe that  $j[\gamma] \in M$ . If we define

$$\mathcal{U} = \{X \subseteq \mathcal{P}_\delta(\gamma) \mid j[\gamma] \in j(X)\},$$

then  $\mathcal{U}$  is a  $\delta$ -complete ultrafilter on  $\mathcal{P}_\delta(\gamma)$ . Moreover,  $\mathcal{U}$  is *normal* in the sense that if  $X \in \mathcal{U}$  and  $f$  is a choice function for  $X$ , then there exists  $Y \in \mathcal{U}$  and  $\alpha < \gamma$  such that  $Y \subseteq X$  and  $f(\sigma) = \alpha$  for every  $\sigma \in Y$ . Equivalently, if  $\langle X_\alpha \mid \alpha < \gamma \rangle$  is a sequence of sets from  $\mathcal{U}$ , then the diagonal intersection,

$$\Delta_{\alpha < \gamma} X_\alpha = \{\sigma \in \mathcal{P}_\delta(\gamma) \mid \sigma \in X_\alpha \text{ for every } \alpha \in \sigma\}$$

also belongs to  $\mathcal{U}$ . In addition,  $\mathcal{U}$  is *fine* in the sense that for every  $\alpha < \gamma$ ,

$$\{\sigma \in \mathcal{P}_\delta(\gamma) \mid \alpha \in \sigma\} \in \mathcal{U}.$$

Suppose, instead, that we are given a  $\delta$ -complete ultrafilter  $\mathcal{U}$  on  $\mathcal{P}_\delta(\gamma)$  which is both normal and fine. Then the ultrapower map derived from  $\mathcal{U}$  can be shown to witness that  $\delta$  is a  $\gamma$ -supercompact cardinal. We might refer to such an ultrafilter (fine, normal and  $\delta$ -complete) as a  $\gamma$ -*supercompactness measure*.

Less well-known is the following characterisation of  $\delta$  being supercompact that more transparently related to extendibility.

**Theorem 12** (Magidor). *A cardinal  $\delta$  is supercompact iff for all  $\kappa > \delta$  and  $a \in V_\kappa$ , there exist  $\bar{\delta} < \bar{\kappa} < \delta$ ,  $\bar{a} \in V_{\bar{\kappa}}$  and an elementary embedding  $j : V_{\bar{\kappa}+1} \rightarrow V_{\kappa+1}$  such that  $\text{crit}(j) = \bar{\delta}$ ,  $j(\bar{\delta}) = \delta$  and  $j(\bar{a}) = a$ .*

*Proof.* First we prove the forward direction. Given  $\kappa$  and  $a$ , let  $\gamma = |V_{\kappa+1}|$  and  $j : V \rightarrow M$  witness that  $\delta$  is a  $\gamma$ -supercompact cardinal. Then

$$j \upharpoonright V_{\kappa+1} \in M$$

and witnesses the following sentence in  $M$ : “There exist  $\bar{\delta} < \bar{\kappa} < j(\delta)$ ,  $\bar{a} \in V_{\bar{\kappa}}$  and an elementary embedding  $i : V_{\bar{\kappa}+1} \rightarrow V_{j(\kappa)+1}$  such that  $\text{crit}(i) = \bar{\delta}$ ,  $i(\bar{\delta}) = j(\delta)$  and  $i(\bar{a}) = j(a)$ .” Since  $j$  is elementary, we are done.

For the reverse direction, let  $\gamma > \delta$  be given. Apply the right side with  $\kappa = \gamma + \omega$ . (The choice of  $a$  is irrelevant.) This yields  $\bar{\kappa}$ ,  $\bar{\delta}$  and  $j$  as specified. Take  $\bar{\gamma}$  such that  $j(\bar{\gamma}) = \gamma$ . Now  $j[\bar{\gamma}] \in V_{\kappa+1}$  so it induces a normal fine ultrafilter  $\bar{\mathcal{U}}$  on  $\mathcal{P}_{\bar{\delta}}(\bar{\gamma})$ . Observe that  $\bar{\mathcal{U}} \in V_{\bar{\gamma}+\omega}$ , so we can define  $\mathcal{U} = j(\bar{\mathcal{U}})$ .

Then, by elementarity,  $V_{\kappa+1}$  believes that  $\mathcal{U}$  is a normal fine ultrafilter on  $\mathcal{P}_\delta(\gamma)$ , and is large enough to bear true witness to such a belief. Thus  $\delta$  is  $\gamma$ -supercompact.  $\square$

We will use the Solovay splitting theorem. A proof can be found in [1] Theorem 8.10 or, using generic embeddings, in [1] Lemma 22.27.

**Theorem 13** (Solovay). *Let  $\gamma$  be a regular uncountable cardinal. Then every stationary subset of  $\gamma$  can be partitioned into  $\gamma$  many stationary sets.*

We will also make key use of the following theorem, which provides a single set that belongs to every  $\gamma$ -supercompactness measure on  $\mathcal{P}_\delta(\gamma)$ . We will refer to this set as the *Solovay set*.

**Theorem 14** (Solovay). *Let  $\delta$  be supercompact and  $\gamma > \delta$  be regular. Then there exists an  $X \subseteq \mathcal{P}_\delta(\gamma)$  such that the sup function is injective on  $X$  and every  $\gamma$ -supercompactness measure contains  $X$ .*

*Proof.* Let  $\langle S_\alpha \mid \alpha < \gamma \rangle$  be a partition of  $\gamma \cap \text{cof}(\omega)$  into stationary sets, which exists by Theorem 13. For  $\beta < \gamma$  such that  $\omega < \text{cf}(\beta) < \delta$ , let  $\sigma_\beta$  be the set of  $\alpha < \beta$  such that  $S_\alpha$  reflects to  $\beta$ . In other words,

$$\sigma_\beta = \{\alpha < \beta \mid S_\alpha \cap \beta \text{ is stationary in } \beta\}.$$

Leave  $\sigma_\beta$  undefined otherwise. Note that it is not possible to partition  $\beta$  into more than  $\text{cf}(\beta)$ -many stationary sets, as can be seen by considering their restrictions to a club in  $\beta$  of order type  $\text{cf}(\beta)$ , so,  $\sigma_\beta \in \mathcal{P}_\delta(\gamma)$ . Define  $X = \{\sigma_\beta \mid \text{sup}(\sigma_\beta) = \beta\}$ . Clearly, the sup function is an injection on  $X$  so given  $\mathcal{U}$  be a normal fine ultrafilter on  $\mathcal{P}_\delta(\gamma)$  it remains to see that  $X \in \mathcal{U}$ . Let  $j : V \rightarrow M$  be the embedding associated to  $\mathcal{U}$ . In fact  $\mathcal{U}$  is the corresponding ultrafilter derived from  $j$ , so what we need to see is that

$$j[\gamma] \in j(X).$$

Let  $\beta = \text{sup}(j[\gamma])$  and

$$\langle S_\alpha^* \mid \alpha < j(\gamma) \rangle = j(\langle S_\alpha \mid \alpha < \gamma \rangle).$$

Clearly,  $\beta < j(\gamma)$  and  $\omega < \text{cf}(\beta) < j(\delta)$ , so we are left to show that

$$j[\gamma] = \{\alpha < \beta \mid M \models S_\alpha^* \cap \beta \text{ is stationary in } \beta\}.$$

First we show containment in the forward direction. Consider any  $\eta < \gamma$ . Then we want  $S_{j(\eta)}^* = j(S_\eta)$  to be stationary. Given  $C$  a club subset of  $\beta$  that belongs to  $M$ , define  $D = \{\alpha < \gamma \mid j(\alpha) \in C\}$ . Because  $j$  is continuous at ordinals of countable cofinality,  $D$  is an  $\omega$ -club in  $\gamma$ . But  $S_\eta$  contains only ordinals of countable cofinality and is stationary in  $\gamma$  so  $S_\eta \cap D \neq \emptyset$ . Hence  $j(S_\eta) \cap C \neq \emptyset$ .

For containment in the reverse direction, consider any  $\alpha < \beta$  such that, in  $M$ ,  $S_\alpha^* \cap \beta$  is stationary in  $\beta$ . Working in  $M$ , as  $j[\gamma]$  is an  $\omega$ -club in  $\beta$  and  $S_\alpha^*$  contains only ordinals of countable cofinality, there exists  $\eta < \gamma$  such that  $j(\eta) \in S_\alpha^*$ . But  $j[\gamma]$  is partitioned by the  $j[S_\theta]$  for  $\theta < \gamma$  so we can take

$\theta < \gamma$  such that  $j(\eta) \in j[S_\theta] \subseteq j(S_\theta) = S_{j(\theta)}^*$ . This means  $S_\alpha^* \cap S_{j(\theta)}^* \neq \emptyset$  so  $\alpha = j(\theta) \in j[\gamma]$ .  $\square$

*Remark.* The proof of Theorem 14 can be easily generalised to prove the following. Assume that  $j : V \rightarrow M$  is a  $\gamma$ -supercompact embedding, where  $\gamma$  is regular. Let  $\kappa < \gamma$  be also regular,  $\beta = \sup j[\gamma]$  and  $\tilde{\beta} = \sup j[\kappa]$ . Then given a partition  $\langle S_\alpha \mid \alpha < \kappa \rangle$  of  $\text{cof}(\omega) \cap \gamma$  into stationary sets, we have that

$$j[\kappa] = \{\alpha \in \tilde{\beta} \mid S_\alpha^* \cap \beta \text{ is stationary in } \beta\},$$

where  $\langle S_\alpha^* \mid \alpha < j(\kappa) \rangle = j(\langle S_\alpha \mid \alpha < \kappa \rangle)$ .

## 5. WEAK EXTENDER MODELS

In inner model theory, the word *extender* has taken on a very general meaning as any object that captures the essence of a given large cardinal property. Sometimes ultrafilters or systems of ultrafilters are used. At other times, elementary embeddings or restrictions of elementary embeddings are more relevant. We have already seen two first-order ways to express supercompactness. An easier example is measurability: if  $U$  is a normal measure on  $\kappa$  and  $j : V \rightarrow M$  is the corresponding ultrapower map, then  $U$  and  $j \upharpoonright V_{\kappa+1}$  carry exactly the same information.

Building a canonical inner model with a supercompact cardinal has been a major open problem in set theory for decades. Canonical inner models for measurable cardinals were produced early on. Letting  $U$  be a normal measure on  $\kappa$  and setting  $\bar{U} = U \cap L[U]$ , we can see that  $\bar{U} \in L[U]$  and  $L[U] \models \bar{U}$  is a normal measure on  $\kappa$ . The general theory of  $L[U]$  does not depend on there being measurable cardinals in  $V$  but this was an important first step.

**Definition 15.** A transitive class  $N$  model of ZFC is called a *weak extender model for  $\delta$  supercompact* iff for every  $\gamma > \delta$  there exists a normal fine measure  $\mathcal{U}$  on  $\mathcal{P}_\delta(\gamma)$  such that

- (1)  $N \cap \mathcal{P}_\delta(\gamma) \in \mathcal{U}$  and
- (2)  $\mathcal{U} \cap N \in N$ .

The first condition says that  $\mathcal{U}$  *concentrates* on  $N$ . In the case of the measurable cardinal, which we discussed above, we get the analogous first condition for free because  $L[U] \cap \kappa = \kappa \in U$ . We might refer to the second condition as saying that  $\mathcal{U}$  is *amenable* to  $N$ .

**Lemma 16.** *If  $N$  is a weak extender model for  $\delta$  supercompact, then it has the  $\delta$ -covering property.*

*Proof.* Note that it is enough to prove  $\delta$ -covering for sets of ordinals. Now, given  $\tau \subseteq \gamma$  with  $|\tau| < \delta$ , let  $\mathcal{U}$  be a  $\gamma$ -supercompactness measure such that  $N \cap \mathcal{P}_\delta(\gamma) \in \mathcal{U}$  and  $\mathcal{U} \cap N \in N$ . By fineness, for each  $\alpha < \gamma$ , we have that  $\{\sigma \in \mathcal{P}_\delta(\gamma) \mid \alpha \in \sigma\} \in \mathcal{U}$ . Hence, as  $|\tau| < \delta$ , by  $\delta$ -completeness we

have  $\{\sigma \in \mathcal{P}_\delta(\gamma) \mid \tau \subseteq \sigma\}$  belongs to  $\mathcal{U}$ . Also as  $N \cap \mathcal{P}_\delta(\gamma) \in \mathcal{U}$ , there is a  $\sigma \in N \cap \mathcal{P}_\delta(\gamma)$  and  $\sigma \supseteq \tau$  as desired.  $\square$

**Lemma 17.** *Suppose  $N$  is a weak extender model for  $\delta$  supercompact and  $\gamma > \delta$  is such that  $N \models$  ‘ $\gamma$  is a regular cardinal’. Then  $|\gamma| = \text{cf}(\gamma)$ .*

*Proof.* Let  $\gamma > \delta$ . Of course,  $\text{cf}(\gamma) \leq |\gamma|$ . Now we prove the reverse inequality. By Lemma 16,  $N$  satisfies the  $\delta$ -covering property so, as  $N$  believes  $\gamma$  is a regular cardinal, we have that  $\text{cf}(\gamma) \geq \delta$ . Now fix  $\mathcal{U}$  a  $\gamma$ -supercompactness measure, such that  $N \cap \mathcal{P}_\delta(\gamma) \in \mathcal{U}$  and  $\mathcal{U} \cap N \in N$ . As  $\gamma$  is a regular cardinal of  $N$ , we may apply Theorem 14 within  $N$  and get a Solovay set  $X \in N$ . So the sup function is an injection on  $X$  and  $X$  belongs to  $\mathcal{U}$ . Now fix a club  $D \subseteq \gamma$  of order type  $\text{cf}(\gamma)$  and define  $A = \{\sigma \in \mathcal{P}_\delta(\gamma) \mid \text{sup}(\sigma) \in D\}$ .

We first claim that  $A \in \mathcal{U}$ . Letting  $j : V \rightarrow M$  be the ultrapower map induced by  $\mathcal{U}$ , it is enough to show that  $j[\gamma] \in j(A)$ . Define  $\beta = \text{sup } j[\gamma]$ . By the definition of  $A$ , we need to see that  $\beta \in j(D)$ . Note that  $j(D)$  is a club in  $j(\gamma)$ , and as  $D$  is unbounded in  $\gamma$  we have that  $j[\gamma] \cap j(D)$  is unbounded in  $\beta$ . Thus  $j(D)$  being closed implies  $\beta \in j(D)$ . Hence  $\{\sigma \in X \mid \text{sup}(\sigma) \in D\} \in \mathcal{U}$ . Recall that  $\mathcal{U}$  is fine, so

$$\gamma = \bigcup \{\sigma \in X \mid \text{sup}(\sigma) \in D\}.$$

Now, because the sup function is injective on  $X$ , we have that the cardinality of  $\gamma$  is at most  $\delta |D|$ . But the order type of  $D$  is  $\text{cf}(\gamma)$ , so  $|\gamma| \leq \delta \text{cf}(\gamma)$ . Finally remember  $\delta \leq \text{cf}(\gamma)$ , so  $|\gamma| \leq \text{cf}(\gamma)$  which concludes the proof.  $\square$

**Corollary 18.** *Let  $N$  be a weak extender model for  $\delta$  supercompact and  $\gamma > \delta$  be a singular cardinal, then*

- (1)  $N \models$  ‘ $\gamma$  is singular’ and
- (2)  $\gamma^+ = (\gamma^+)^N$ .

*Proof.* Immediate by Lemma 17.  $\square$

Next, we characterise the HOD Conjecture in two ways, each of which says HOD is close to  $V$  in a certain sense.

**Theorem 19.** *Let  $\delta$  be an extendible cardinal. The following are equivalent.*

- (1) *The HOD Conjecture.*
- (2) *HOD is a weak extender model for  $\delta$  supercompact.*
- (3) *Every singular cardinal  $\gamma > \delta$ , is singular in HOD and  $\gamma^+ = (\gamma^+)^{\text{HOD}}$ .*

*Proof.* (2) implies (3) is just Lemma 18. That (3) implies (1) was shown in the discussion right after the definition of the HOD Conjecture (Definition 11). We now prove (1) implies (2).

Given  $\zeta > \delta$ , we wish to show that there is a  $\zeta$ -supercompactness measure  $\mathcal{U}$  such that  $\mathcal{U} \cap \text{HOD} \in \text{HOD}$  and  $\mathcal{P}_\delta(\gamma) \cap \text{HOD} \in \mathcal{U}$ . For this, take  $\gamma > 2^\zeta$ , such that  $|V_\gamma|^{\text{HOD}} = \gamma$  and fix a regular cardinal  $\lambda > 2^\gamma$  such that  $\lambda$  is not  $\omega$ -strongly measurable in HOD. Finally, pick  $\eta > \lambda$  such that the

defining formula for HOD is absolute for  $V_\eta$ , whence  $\text{HOD}^{V_\eta} = \text{HOD} \cap V_\eta$ . As  $\delta$  is extendible, there is an elementary embedding  $j : V_{\eta+1} \rightarrow V_{j(\eta)+1}$  with critical point  $\delta$ .

**Claim.**  $j[\gamma] \in \text{HOD}^{V_{j(\eta)}}$ .

As  $\lambda$  is not  $\omega$ -strongly measurable in HOD and  $2^\gamma < \lambda$  (in  $V$  and so in HOD) there is a partition  $\langle S_\alpha \mid \alpha \in \gamma \rangle$  of  $\text{cof}(\omega) \cap \lambda$  into stationary sets such that  $\langle S_\alpha \mid \alpha < \gamma \rangle \in \text{HOD}$ . Thus  $\langle S_\alpha \mid \alpha \in \gamma \rangle \in \text{HOD}^{V_\eta}$ . By the elementarity of  $j$  we have

$$\langle S_\alpha^* \mid \alpha \in j(\gamma) \rangle = j(\langle S_\alpha \mid \alpha \in \gamma \rangle) \in \text{HOD}^{V_{j(\eta)}}.$$

Let  $\beta = \sup j[\lambda]$  and  $\tilde{\beta} = \sup j[\gamma]$ . By the remark after the proof of Theorem 14,

$$j[\gamma] = \{\alpha \in \tilde{\beta} \mid S_\alpha^* \cap \beta \text{ is stationary in } \beta\}$$

This shows that  $j[\gamma]$  is OD in  $V_{j(\eta)}$ . Moreover  $V_{j(\eta)}$  is correct about stationarity in  $\beta$ , thus  $j[\gamma] \in \text{HOD}^{V_{j(\eta)}}$ . Also note that  $j[\zeta] \in \text{HOD}^{V_{j(\eta)}}$ .

Now, observe that  $\text{HOD}^{V_{j(\eta)}} \subset \text{HOD}$ , so we have that  $j[\gamma] \in \text{HOD}$ . Also  $|V_\gamma|^{\text{HOD}} = \gamma$ , so we may take  $e \in \text{HOD}$  a bijection from  $\gamma$  to  $V_\gamma^{\text{HOD}}$ . Clearly  $j(e)[j[\gamma]] = j[V_\gamma \cap \text{HOD}]$  and so  $j[V_\gamma \cap \text{HOD}] \in \text{HOD}$ . Furthermore, as

$$j \upharpoonright (V_\gamma \cap \text{HOD})$$

is the inverse of the Mostowski collapse, we have that

$$j \upharpoonright (V_\gamma \cap \text{HOD}) \in \text{HOD}.$$

Now, let  $\mathcal{U}$  be the ultrafilter on  $\mathcal{P}_\delta(\zeta)$  derived from  $j$ . That is, for  $A \subseteq \mathcal{P}_\delta(\zeta)$ ,  $A \in \mathcal{U}$  iff  $j[\zeta] \in j(A)$ . So,

$$\mathcal{P}_\delta(\zeta) \cap \text{HOD} \in \mathcal{U} \text{ as } j[\zeta] \in \text{HOD}^{V_{j(\eta)}} = j(\text{HOD} \cap V_\eta)$$

$$\mathcal{U} \cap \text{HOD} \in \text{HOD} \text{ as } j \upharpoonright (V_\gamma \cap \text{HOD}) \in \text{HOD} \text{ and } \gamma > 2^\zeta.$$

Thus  $\mathcal{U}$  concentrates on HOD and is amenable to HOD as desired.  $\square$

As a corollary, we obtain the following version of the HOD Dichotomy, Theorem 2.

**Corollary 20.** *Let  $\delta$  be an extendible cardinal. Then exactly one of the following holds.*

- (1) *For every singular cardinal  $\gamma > \delta$ ,  $\gamma$  is singular in HOD and  $\gamma^+ = (\gamma^+)^{\text{HOD}}$ .*
- (2) *There exists a  $\kappa > \delta$  such that every regular  $\gamma > \kappa$  is measurable in HOD.*

*Proof.* Suppose (2) does not hold, then there are arbitrarily large regular cardinals that are not measurable in HOD. By Lemma 10 there are arbitrarily large regular cardinals that are not  $\omega$ -strongly measurable in HOD. Now by the proof of Theorem 19, this implies that HOD is a weak extender model for  $\delta$  supercompact. Finally Corollary 18 yields (1).  $\square$

## 6. ELEMENTARY EMBEDDINGS OF WEAK EXTENDER MODELS

We now give more evidence that if  $N$  is weak extender model for  $\delta$  supercompact then it is close to  $V$ . We will prove that if  $\delta$  an extendible cardinal,  $N$  is a weak extender model for  $\delta$  supercompact and  $j$  is an elementary embedding between levels of  $N$  with  $\text{crit}(j) \geq \delta$ , then  $j \in N$ . This implies that if  $\delta$  is extendible and the HOD Conjecture holds then there are no elementary embeddings from HOD to HOD with critical point greater or equal  $\delta$ . This says that a natural analog of  $0^\#$  for HOD does not exist. As one would expect from Magidor's characterisation of supercompactness, Theorem 12, there is an alternative formulation of "weak extender model for  $\delta$  supercompact" in terms of suitable elementary embeddings  $j : V_{\bar{\kappa}+1} \rightarrow V_{\kappa+1}$  for  $\bar{\kappa} < \delta$ .

**Theorem 21.** *Let  $N$  be a proper class model of ZFC. Then the following are equivalent:*

- (1)  $N$  is a weak extender model for  $\delta$  supercompact.
- (2) For every  $\kappa > \delta$  and  $b \in V_\kappa$ , there exist two cardinals  $\bar{\kappa}$  and  $\bar{\delta}$  below  $\delta$ ,  $\bar{b} \in V_{\bar{\kappa}}$  and  $j : V_{\bar{\kappa}+1} \rightarrow V_{\kappa+1}$  such that:

$$\text{crit}(j) = \bar{\delta}, \quad j(\bar{\delta}) = \delta, \quad j(\bar{b}) = b,$$

$$j(N \cap V_{\bar{\kappa}}) = N \cap V_\kappa \text{ and}$$

$$j \upharpoonright (V_{\bar{\kappa}} \cap N) \in N.$$

*Proof (2) implies (1).* Given  $\gamma > \delta$ , we may assume  $\gamma = |V_\gamma|$ . Let  $\bar{\kappa} = \gamma + \omega$ . We obtain  $\bar{\kappa}$ ,  $\bar{\delta}$  and  $j$  using (2). Take  $\bar{\gamma}$  such that  $\bar{\kappa} = \bar{\gamma} + \omega$ , whence  $j(\bar{\gamma}) = \gamma$ . Let  $\bar{\mathcal{U}}$  be the measure on  $\mathcal{P}_{\bar{\delta}}(\bar{\gamma})$  derived from  $j$ . That is, for  $A \in \mathcal{P}_{\bar{\delta}}(\bar{\gamma})$

$$A \in \bar{\mathcal{U}} \iff j[\bar{\gamma}] \in j(A).$$

Define  $\mathcal{U} = j(\bar{\mathcal{U}})$ . We show that  $\mathcal{U}$  is a  $\gamma$ -supercompactness measure such that  $\mathcal{P}_\delta(\gamma) \cap N \in \mathcal{U}$  and  $\mathcal{U} \cap N \in N$ .

We claim that  $\mathcal{P}_{\bar{\delta}}(\bar{\gamma}) \cap N \in \bar{\mathcal{U}}$ . By (2), we know that  $j(N \cap V_{\bar{\kappa}}) = N \cap V_\kappa$ . Thus for every  $a \in V_{\bar{\kappa}}$  we have

$$j(a \cap N) = j(a) \cap j(N \cap V_{\bar{\kappa}}) = j(a) \cap N.$$

Recalling that  $\bar{\kappa} = \bar{\gamma} + \omega$ ,

$$j(\mathcal{P}_{\bar{\delta}}(\bar{\gamma}) \cap N) = \mathcal{P}_\delta(\gamma) \cap N.$$

Now, as  $j \upharpoonright (N \cap V_{\bar{\kappa}}) \in N$ , we have  $j[\bar{\gamma}] \in N$ , so  $j[\bar{\gamma}] \in \mathcal{P}_\delta(\gamma) \cap N = j(\mathcal{P}_{\bar{\delta}}(\bar{\gamma}) \cap N)$ , which readily implies our claim.

Finally, by elementarity of  $j$ , we have that  $\mathcal{U}$  is a fine and normal measure on  $\mathcal{P}_\delta(\gamma)$  and, by the previous claim,  $j(N \cap \mathcal{P}_{\bar{\delta}}(\bar{\gamma})) \in \mathcal{U}$ . It follows that

$$N \cap \mathcal{P}_\delta(\gamma) \in \mathcal{U}.$$

Moreover, as  $j \upharpoonright (N \cap V_{\bar{\kappa}}) \in N$ , we have that  $j(\bar{\mathcal{U}} \cap N) \in N$ . Hence

$$j(\bar{\mathcal{U}} \cap N) = j(\bar{\mathcal{U}}) \cap N = \mathcal{U} \cap N \in N.$$

This concludes the first direction. □

*Proof (1) implies (2).* Let  $\kappa > \delta$ ,  $b \in V_\kappa$  and fix  $\gamma > |V_{\kappa+\omega}|$  such that  $|V_\gamma|^N = \gamma$ . Fix a  $\gamma$ -supercompactness measure  $\mathcal{U}$  such that  $\mathcal{P}_\delta(\gamma) \cap N \in \mathcal{U}$  and  $\mathcal{U} \cap N \in N$ . Now, fix a bijection  $e : \gamma \rightarrow V_\gamma^N$  in  $N$ . We now work in  $N$ . Define  $N_\sigma$  to be the Mostowski collapse of  $e[\sigma]$ , and

$$Y = \{\sigma \in N \cap \mathcal{P}_\delta(\gamma) \mid N_\sigma = V_{\text{otp}(\sigma)}^N\}.$$

Hence  $Y$  is a club of  $N \cap \mathcal{P}_\delta(\gamma)$ , so it belongs to  $\mathcal{U} \cap N$ . Thus if  $j : V \rightarrow M$  is the ultrapower map it follows that  $j[\gamma] \in j(Y)$ . This implies from the definition of  $Y$  that the collapse of  $j(e)[j[\gamma]]$  is exactly  $V_\gamma \cap j(N \cap V_\delta)$ . Note also that  $j(e)[j[\gamma]] = j[V_\gamma \cap N]$ . Of course  $V_\gamma \cap N$  is the collapse of  $j[V_\gamma \cap N]$ , so

$$V_\gamma \cap N = V_\gamma \cap j(N \cap V_\delta),$$

which implies

$$V_\kappa \cap N = V_\kappa \cap j(N \cap V_\delta).$$

It is clear that for  $\sigma \in N$  we have that  $e[\sigma] \in V_{\gamma+1} \cap N$ . Then by Los' Theorem we have that  $j[V_\gamma \cap N] = j(e)[j[\gamma]] \in j(V_{\gamma+1} \cap N)$ . Notice that the collapsing map of  $j[V_\gamma \cap N]$  is just the inverse of  $j \upharpoonright (V_\gamma \cap N)$ , thus

$$j \upharpoonright (V_\kappa \cap N) \in j(V_{\gamma+1} \cap N).$$

Now  $M$  being closed under  $\gamma$  sequences and  $\gamma > |V_{\kappa+\omega}|$  imply  $j \upharpoonright (V_{\kappa+1})$  belongs to  $M$ . Working in  $M$  let  $i = j \upharpoonright (V_{\kappa+1})$ . Now let us prove that the two previous equations imply that  $i$  satisfy the conditions of (2) relative to  $j(\kappa)$ ,  $j(b)$  and  $j(N \cap V_{\gamma+1})$  in  $M$ . Indeed the equations give

$$i \upharpoonright (V_\kappa \cap j(N \cap V_{\gamma+1})) = i \upharpoonright (V_\kappa \cap N) \in N \cap j(V_{\gamma+1}).$$

Furthermore as  $i$  and  $j$  agree,

$$i(V_\kappa \cap j(N \cap V_{\gamma+1})) = i(V_\kappa \cap N) = j(N \cap V_{\gamma+1}) \cap V_{j(\kappa)}.$$

Also  $j(b) = i(b)$ , so by elementarity (2) holds in  $V$  with respect to  $\kappa$ ,  $b$  and  $N$ . □

We now prove that if  $\delta$  is an extendible cardinal and  $N$  is a weak extender model for  $\delta$  supercompact, then  $N$  sees all elementary embeddings between its levels.

**Theorem 22.** *Let  $\delta$  be an extendible cardinal. Assume that  $N$  is a weak extender model for  $\delta$  supercompact and  $\gamma > \delta$  is a cardinal in  $N$ . Let*

$$j : H(\gamma^+)^N \rightarrow H(j(\gamma)^+)^N$$

*be an elementary embedding with  $\delta \leq \text{crit}(j)$  and  $j \neq \text{id}$ . Then  $j \in N$ .*

*Proof.* Define  $b = (j, \gamma)$  and let  $\kappa$  be a cardinal much larger than  $j(\gamma)$ . Now, as  $N$  is a weak extender model for  $\delta$  supercompact, we may apply Theorem 21 to  $\kappa$  and  $b$ . Hence, we get an elementary embedding  $\pi : V_{\bar{\kappa}+1} \rightarrow V_{\kappa+1}$ , two ordinals  $\bar{\delta}$ ,  $\bar{\gamma}$  and  $\bar{j} \in V_{\bar{\kappa}}$ , with the following properties

$$j(N \cap V_{\bar{\kappa}}) = N \cap V_\kappa, \pi \upharpoonright (V_{\bar{\kappa}} \cap N) \in N$$

and

$$\text{crit}(\pi) = \bar{\delta}, \pi(\bar{j}) = j, \pi(\bar{\delta}) = \delta, \pi(\bar{\gamma}) = \gamma, \bar{\kappa} < \delta.$$

Hence, by the elementarity of  $\pi$ , we have that  $\bar{j} : H(\bar{\gamma}^+)^N \rightarrow H(\bar{j}(\bar{\gamma})^+)^N$  is an elementary map with  $\bar{\delta} \leq \text{crit}(\bar{j})$ . Furthermore as  $\bar{\kappa}$  is very large above  $\bar{j}(\bar{\gamma})$ , we have that

$$\pi \upharpoonright (H(\bar{j}(\bar{\gamma})^+)^N) \in N,$$

hence  $\pi \upharpoonright (H(\bar{j}(\bar{\gamma})^+)^N) \in H(\gamma^+)^N$ . Define  $\pi^* = j(\pi \upharpoonright (H(\bar{j}(\bar{\gamma})^+)^N))$ . Now we wish to show that  $\bar{j} \in N$ . This will be done by proving that  $N$  can actually compute  $\bar{j}$ . For this, take  $\bar{a} \in H(\bar{\gamma}^+)^N$  and  $\bar{s} \in H(\bar{j}(\bar{\gamma})^+)^N$ . Let  $\pi(\bar{a}) = a$  and  $\pi(\bar{s}) = s$  then,

$$\begin{aligned} \bar{s} \in \bar{j}(\bar{a}) &\iff s \in j(a) \\ &\iff s \in j(\pi(\bar{a})) \\ &\iff \pi(\bar{s}) \in j(\pi \upharpoonright (H(\bar{j}(\bar{\gamma})^+)^N)(\bar{a})) \\ &\iff \pi(\bar{s}) \in \pi^*(j(\bar{a})) \\ &\iff \pi \upharpoonright (H(\bar{j}(\bar{\gamma})^+)^N)(\bar{s}) \in \pi^*(\bar{a}). \end{aligned}$$

Where the last equivalence follows because  $\text{crit}(j) > \bar{\kappa}$  and  $\bar{\kappa}$  is sufficiently large above  $\bar{j}(\bar{\gamma})^N$ . Now as  $\pi^*$  and  $\pi \upharpoonright (H(\bar{j}(\bar{\gamma})^+)^N)$  are in  $N$ ,  $\bar{j} \in N \cap V_{\bar{\kappa}}$ . Since  $\pi$  stretches  $N$  correctly up to rank  $\bar{\kappa}$ , we conclude  $j = \pi(\bar{j}) \in N$  as desired.  $\square$

Now we show that, if  $\delta$  is an extendible cardinal, then no elementary embedding maps a weak extender model for  $\delta$  supercompact to itself. For this we recall the following form of Kunen's theorem.

**Theorem 23** (Kunen). *Let  $\kappa$  be an ordinal. Then there is no non-trivial elementary embedding*

$$i : V_{\kappa+2} \rightarrow V_{\kappa+2}.$$

The proof can be found in [2] Theorem 23.14.

**Theorem 24.** *Let  $N$  be a weak extender model for  $\delta$  supercompact. Then there is no elementary embedding  $j : N \rightarrow N$  with  $\delta \leq \text{crit}(j)$  and  $j \neq \text{id}$ .*

*Proof.* Suppose for contradiction that there is such a  $j$ . Let  $\kappa > \delta$  be a fixed point of  $j$ . Then the restriction of  $j$  to  $V_{\kappa+2}^N$  is the an elementary embedding  $i : V_{\kappa+2}^N \rightarrow V_{\kappa+2}^N$  with  $\text{crit}(i) \geq \delta$ . Theorem 22 implies  $i \in N$ . This contradicts Theorem 23 within  $N$ .  $\square$

**Corollary 25.** *Assume the HOD Conjecture. If  $\delta$  is an extendible cardinal, then there is no  $j : \text{HOD} \rightarrow \text{HOD}$  with  $\delta \leq \text{crit}(j)$  and  $j \neq \text{id}$ .*

*Proof.* Follows from the Theorem 24 and Theorem 19.  $\square$

Finally we give an example  $N$  of a weak extender model for  $\delta$  supercompact other than  $V$ .  $N$  will be such that there is a nontrivial elementary embedding  $j : N \rightarrow N$ , with  $\text{crit}(j) < \delta$ . The point of the next example is



that actually one can have a weak extender model for  $\delta$  supercompact but it lacks structural properties, such as the ones HOD and  $L$  possess. Note that this makes Theorem 24 actually optimal.

For the example we will use the following fact.

**Lemma 26.** *Let  $\kappa$  be a measurable cardinal,  $\mu$  a measure on  $\kappa$  and  $j : V \rightarrow M$  the ultrapower map given by  $\mu$ . Also let  $\nu$  be a  $\delta$ -complete measure, for some  $\delta > \kappa$ ,  $k : V \rightarrow N$  the ultrapower map given by  $\nu$  and  $l : M \rightarrow \text{Ult}(M, j(\nu))$  the ultrapower map. Then  $k \upharpoonright M = l$ .*

*Proof.* First, observe that as the critical point of  $k$  is above  $\kappa$ , then  $\mu \in N$ . Apply  $\mu$  to  $N$  and let  $j' : N \rightarrow \text{Ult}(N, \mu)$  be the ultrapower map.  ${}^\delta N \subset N$  because  $\nu$  is  $\delta$ -complete, so all functions from  $\kappa$  to  $N$  are in  $N$ , and this readily implies  $j \upharpoonright N = j'$ .

Now, for  $j(f)(\kappa)$  an element of  $M$ , we wish to see that  $k(j(f)(\kappa)) = l(j(f)(\kappa))$ . For simplicity,  $j$  for a restriction of  $j$  to a suitable rank-initial segment which can then be treated as an element; likewise for  $k$ . By elementarity we have that  $k(j(f)(\kappa)) = k(j)(k(f))(k(\kappa))$ , but  $k(j)$  is  $j'$  which is the restriction of  $j$  to  $N$  so,

$$\begin{aligned} k(j(f)(\kappa)) &= j(k(f))(\kappa) \\ &= j(k)(j(f))(\kappa) \\ &= l(j(f))(\kappa) \\ &= l(j(f))(l(\kappa)) \\ &= l(j(f)(\kappa)) \end{aligned}$$

In other words,  $k$  restricts to  $l$  as desired.  $\square$

**Example 27.** Let  $\delta$  be a supercompact cardinal. Then there is  $N$  a weak extender model for  $\delta$  supercompact, and a nontrivial  $j : N \rightarrow N$  with  $\text{crit}(j) < \delta$ .

Let  $\kappa < \delta$  be a measurable cardinal and take  $\mu$  a measure on  $\kappa$ . Let

$$V = M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow \cdots \rightarrow M_\omega$$

be the internal iteration of  $V$  by  $\mu$  of length  $\omega$ . So we have  $M_0 = V$ ,  $\kappa_0 = \kappa$ ; and inductively for naturals  $n > 0$  define  $\mu_n = i_{n-1, n}(\mu_{n-1})$ ,  $\kappa_n = i_{n-1}(\kappa_{n-1}) = \text{crit}(\mu_n)$  and let  $i_{n, n+1} : M_n \rightarrow M_{n+1}$  be the map induced by taking the ultrapower of  $M_n$  by  $\mu_n$ .  $M_\omega$  is then the direct limit of the system and  $i_{n, \omega} : M_n \rightarrow M_\omega$  the induced embeddings.  $M_\omega$  is well founded and so we identify it with its transitive collapse (see Theorem 19.7 of [1]). Define  $N = M_\omega$ .

Now, we show that  $N$  is a weak extender model for  $\delta$  supercompact. This is equivalent to showing that for unboundedly many  $\gamma$  there is a  $\gamma$ -supercompactness measure that concentrates on  $N$  and is amenable to  $N$ . Note that  $i_{0, \omega}(\delta) = \delta$  and that for unboundedly many ordinals  $\gamma$ , we have that  $i_{0, \omega}(\gamma) = \gamma$ . Fix such  $\gamma$  and let  $\mathcal{U}$  be a normal and fine measure on

$\mathcal{P}_\delta(\gamma)$ . We prove that  $\mathcal{U}$  is a suitable measure for  $N$ . Now, let  $\mathcal{W} = i_{0,\omega}(\mathcal{U})$ , and  $\mathcal{W}_n = i_{0,n}(\mathcal{U})$  (observe that, for each  $n$ ,  $\mathcal{W}_n$  is a normal fine measure on  $\mathcal{P}_\delta(\gamma)$  in  $M_n$ ). By Lemma 26 the map induced by taking  $\text{Ult}(V, \mathcal{U})$  restricts to the one given by  $\text{Ult}(M_1, \mathcal{W})$ . Inductively we have that if  $k_n : M_n \rightarrow \text{Ult}(V, \mathcal{W}_n)$  is the ultrapower map, then  $k_n = k_0 \upharpoonright M_n$ . It follows then that as  $i_{n,\omega}(\mathcal{W}_n) = \mathcal{W}$  (for each  $n$ ) and  $N$  is the direct limit of the initial system, we have that  $k = k_0 \upharpoonright N$ , where  $k : N \rightarrow \text{Ult}(N, \mathcal{W})$  is the ultrapower map given by  $\mathcal{W}$ . Therefore for  $A \in N$ ,  $k[\gamma] \in k(A)$  iff  $k_0[\gamma] = k_0(A)$ , which readily implies  $\mathcal{W} = \mathcal{U} \cap N$ ; in other words  $\mathcal{U}$  is amenable to  $N$ . Also,  $\mathcal{U}$  concentrates on  $N$  as  $N \cap \mathcal{P}_\delta(\gamma) \in \mathcal{W} \subseteq \mathcal{U}$ , as desired. Thus  $N$  is a weak extender model for  $\delta$  supercompact.

Finally, observe that if  $j = i_{0,1} \upharpoonright N$  then  $j : N \rightarrow N$  as  $N$  is the  $\omega$ -th iterate, so we have a nontrivial embedding from  $N$  to  $N$ , the key point here is that  $\text{crit}(j) < \delta$ .

## 7. CONSEQUENCES OF THE HOD CONJECTURE

We conclude by summarising without proof some results that would follow if the HOD Conjecture were proved to be a theorem of ZFC.

**Theorem 28 (ZF).** *Assume that ZFC proves the HOD Conjecture. Suppose  $\delta$  is an extendible cardinal. Then there is a transitive class  $M \subseteq V$  such that:*

- (1)  $M \models \text{ZFC}$
- (2)  $M$  is  $\Sigma_2(a)$ -definable for some  $a \in V_\delta$
- (3) Every set of ordinals is  $< \delta$ -generic over  $M$
- (4)  $M \models$  “ $\delta$  is an extendible cardinal”

The conclusion of the theorem is that there is an inner model  $M$  which is both close to  $V$  and in which the Axiom of Choice holds. (See [4] Theorem 229 for proof of a stronger result.) This is close to “proving” the Axiom of Choice from large cardinal axioms and suggests the following conjecture.

**Definition 29.** The *Axiom of Choice Conjecture* asserts in ZF, that if  $\delta$  is an extendible cardinal then the Axiom of Choice holds in  $V[G]$ , where  $G$  is  $V$ -generic for collapsing  $V_\delta$  to be countable.

One application of Theorem 28 is the following theorem. (See [4] Theorem 228 for a proof.)

**Theorem 30 (ZF).** *Assume that ZFC proves the HOD Conjecture. Suppose  $\delta$  is an extendible cardinal. Then for all  $\lambda > \delta$  there is no non-trivial elementary embedding  $j : V_{\lambda+2} \rightarrow V_{\lambda+2}$ .*

Thus (assuming that ZFC proves the HOD Conjecture) one nearly has a proof of Kunen’s Theorem (23) without using the Axiom of Choice.

For our final theorem we need a new definition.  $L(\mathcal{P}(OR))$  is built in the same way as the usual  $L$ -hierarchy but allowing the use of all sets of

ordinals in definitions. So it is the least model of ZF that contains all sets of ordinals. Note that, under ZF, this is not necessarily the whole of  $V$ .

**Theorem 31** (ZF). *Assume that ZFC proves the HOD Conjecture. Suppose that  $\delta$  is an extendible cardinal. Then in  $L(\mathcal{P}(OR))$ :*

- (1)  $\delta$  is an extendible cardinal.
- (2) The Axiom of Choice Conjecture holds.

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