COUNTABLE BOREL EQUIVALENCE RELATIONS

SIMON THOMAS AND SCOTT SCHNEIDER

Introduction. These notes are based upon a day-long lecture workshop presented by Simon Thomas at the University of Ohio at Athens on November 17, 2007. The workshop served as an intensive introduction to the emerging theory of countable Borel equivalence relations. These notes are an updated and slightly expanded version of an earlier draft which was compiled from the lecture slides by Scott Schneider.

1. First Session

1.1. **Standard Borel Spaces and Borel Equivalence Relations.** A topological space is said to be *Polish* if it admits a complete, separable metric. If $\mathcal{B}$ is a σ-algebra of subsets of a given set X, then the pair $(X, \mathcal{B})$ is called a *standard Borel space* if there exists a Polish topology $\mathcal{T}$ on X that generates $\mathcal{B}$ as its Borel σ-algebra; in which case, we write $\mathcal{B} = \mathcal{B}(\mathcal{T})$. For example, each of the sets $\mathbb{R}$, $[0,1]$, $\mathbb{N}^{\mathbb{N}}$, and $2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$ is Polish in its natural topology, and so may be viewed, equipped with its corresponding Borel structure, as a standard Borel space.

The abstraction involved in passing from a topology to its associated Borel structure is analagous to that of passing from a metric to its induced topology. Just as distinct metrics on a space may induce the same topology, distinct topologies may very well generate the same Borel σ-algebra. In a standard Borel space, then, one "remembers" only the Borel sets, and forgets which of them were open; it is natural therefore to imagine that *any* of them might have been, and indeed this is the case:

Theorem 1.1.1. *Let $(X, \mathcal{T})$ be a Polish space and $Y \subseteq X$ any Borel subset. Then there exists a Polish topology $\mathcal{T}_Y \supseteq \mathcal{T}$ such that $\mathcal{B}(\mathcal{T}_Y) = \mathcal{B}(\mathcal{T})$ and Y is clopen in $(X, \mathcal{T}_Y)$.*

It follows that if $(X, \mathcal{B})$ is a standard Borel space and $Y \in \mathcal{B}$, then $(Y, \mathcal{B} \upharpoonright Y)$ is also a standard Borel space. In fact, so much structural information is "forgotten" in passing from a Polish space to its Borel structure that we obtain the following theorem of Kuratowski [22].

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Theorem 1.1.2. *There exists a unique uncountable standard Borel space up to isomorphism.*

It turns out that many classification problems from diverse areas of mathematics may be viewed as definable equivalence relations on standard Borel spaces. For example, consider the problem of classifying all countable graphs up to graph isomorphism. Let $\mathcal{C}$ be the set of graphs of the form $\Gamma = \langle \mathbb{N}, E \rangle$. Then identifying each graph $\Gamma \in \mathcal{C}$ with its edge relation $E \in 2^{\mathbb{N}^2}$, one easily checks that $\mathcal{C}$ is a Borel subset of $2^{\mathbb{N}^2}$ and hence is a standard Borel space. Moreover, the isomorphism relation on $\mathcal{C}$ is simply the orbit equivalence relation arising from the natural action of $\mathrm{Sym}(\mathbb{N})$ on $\mathcal{C}$. More generally, if σ is a sentence of $\mathcal{L}_{\omega_1,\omega}$, then

$$\mathrm{Mod}(\sigma) = \{\, \mathcal{M} = \langle \mathbb{N}, \cdots \rangle \mid \mathcal{M} \models \sigma \,\}$$

is a standard Borel space, and the isomorphism relation on $\mathrm{Mod}(\sigma)$ is the orbit equivalence relation arising from the natural $\mathrm{Sym}(\mathbb{N})$-action. However, while the isomorphism relation on $\mathrm{Mod}(\sigma)$ is always an analytic subset of $\mathrm{Mod}(\sigma) \times \mathrm{Mod}(\sigma)$, it is not in general a Borel subset; for example, the graph isomorphism relation on $\mathcal{C}$ is not Borel. On the other hand, the restriction of graph isomorphism to the standard Borel space of connected locally finite graphs is Borel; and more generally, the isomorphism relation on a standard Borel space of countable structures will be Borel if each structure is "finitely generated" in some broad sense. With these examples in mind, we make the following definitions.

Definition 1.1.3. *If X is a standard Borel space, then a* Borel equivalence relation *on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2.*

Definition 1.1.4. *If G is a Polish group, then a* standard Borel G-space *is a standard Borel space X equipped with a Borel G-action $(g, x) \mapsto g \cdot x$. The corresponding G-orbit equivalence relation is denoted by E_G^X.*

We observe that if G is a countable group and X is a standard Borel G-space, then E_G^X is a Borel equivalence relation. As further examples, we will next consider the standard Borel space $R(\mathbb{Q}^n)$ of torsion-free abelian groups of rank n and the Polish space $\mathcal{G}$ of finitely generated groups.

For each $n \geq 1$, let $\mathbb{Q}^n = \bigoplus_{1 \leq i \leq n} \mathbb{Q}$. Then the standard Borel space of torsion-free abelian groups of rank n is defined to be

$$R(\mathbb{Q}^n) = \{A \leq \mathbb{Q}^n \mid A \text{ contains a basis of } \mathbb{Q}^n\}.$$

Notice that if $A, B \in R(\mathbb{Q}^n)$, then we have that

$$A \cong B \quad \text{iff} \quad \text{there exists } \varphi \in GL_n(\mathbb{Q}) \text{ such that } \varphi(A) = B,$$

and hence the isomorphism relation on $R(\mathbb{Q}^n)$ is the Borel equivalence relation arising from the natural action of $GL_n(\mathbb{Q}^n)$ on $R(\mathbb{Q}^n)$.

As a step towards defining the Polish space $\mathcal{G}$ of finitely generated groups, for each $m \in \mathbb{N}$, let $\mathbb{F}_m$ be the free group on the m generators $\{x_1, \dots, x_m\}$ and let $2^{\mathbb{F}_2}$ be the compact space of all functions $\varphi : \mathbb{F}_2 \to 2$. Then, identifying each subset $S \subseteq \mathbb{F}_m$ with its characteristic function $\chi_S \in 2^{\mathbb{F}_2}$, it is easily checked that the collection $\mathcal{G}_m$ of normal subgroups of $\mathbb{F}_m$ is a closed subset of $2^{\mathbb{F}_2}$. In particular, $\mathcal{G}_m$ is a compact Polish space. Next, as each m-generator group can be realized as a quotient $\mathbb{F}_m/N$ for some $N \in \mathcal{G}_m$, we can identify $\mathcal{G}_m$ with the space of m-generator groups. Finally, there exists a natural embedding $\mathcal{G}_m \hookrightarrow \mathcal{G}_{m+1}$ defined by

$$N \mapsto \text{ the normal closure of } N \cup \{\, x_{m+1} \,\} \text{ in } \mathbb{F}_{m+1};$$

and so we can define the space of finitely generated groups by $\mathcal{G} = \bigcup_{m \geq 1} \mathcal{G}_m$.

By a theorem of Tietze, if $N, M \in \mathcal{G}_m$, then $\mathbb{F}_m/N \cong \mathbb{F}_m/M$ if and only if there exists $\pi \in \operatorname{Aut}(\mathbb{F}_{2m})$ such that $\pi(N) = M$.[1] In particular, it follows that the isomorphism relation $\cong$ on the space $\mathcal{G}$ of finitely generated groups is the orbit equivalence relation arising from the action of the countable group $\operatorname{Aut}_f(\mathbb{F}_\infty)$ of finitary automorphisms of the free group $\mathbb{F}_\infty$ on $\{x_1, x_2, \cdots, x_m, \cdots\}$. (For more details, see either Champetier [6] or Thomas [35].)

1.2. **Borel Reducibility.** We have seen that many naturally occurring classification problems may be viewed as Borel equivalence relations on standard Borel spaces. In particular, the complexity of the problem of finding complete invariants for such classification problems can be measured to some extent by the "structural complexity" of the associated Borel equivalence relations. Here the crucial notion of comparison is that of a *Borel reduction.*

Definition 1.2.1. *If E and F are Borel equivalence relations on the standard Borel spaces X, Y respectively, then we say that E is* Borel reducible *to F, and write $E \leq_B F$, if there exists a Borel map $f : X \to Y$ such that $xEy \leftrightarrow f(x)Ff(y)$. Such a map is called a* Borel

[1] It is probably worth pointing out that this is *not* a misprint. For example, by Dunwoody-Pietrowski [9], there exist normal subgroups N, $M \leqslant \mathbb{F}_2$ with $\mathbb{F}_2/N \cong \mathbb{F}_2/M$ such that $\theta(N) \neq M$ for all $\theta \in \operatorname{Aut}(\mathbb{F}_2)$. However, if we identify N, M with the corresponding normal subgroups of $\mathbb{F}_4$ via the natural embedding $\mathcal{G}_2 \hookrightarrow \mathcal{G}_3 \hookrightarrow \mathcal{G}_4$, then there exists $\pi \in \operatorname{Aut}(\mathbb{F}_4)$ such that $\pi(N) = M$.

reduction *from E to F. We say that E and F are* Borel bireducible*, and write $E \sim_B F$, if both $E \leq_B F$ and $F \leq_B E$; and we write $E <_B F$ if both $E \leq_B F$ and $F \nleq_B E$.*

If E and F are Borel equivalence relations, then we interpret $E \leq_B F$ to mean that the classification problem associated with E is at most as complicated as that associated with F, in the sense that an assignment of complete invariants for F would, via composition with the Borel reduction from E to F, yield one for E as well. Additionally we observe that if $f : E \leq_B F$, then the induced map $\tilde{f} : X/E \to Y/F$ is an embedding of quotient spaces, the existence of which is sometimes interpreted as saying that X/E has "Borel cardinality" less than or equal to that of Y/F.

This notion of Borel reducibility imposes a partial (pre)-order on the collection of Borel equivalence relations, and much of the work currently taking place in the theory of Borel equivalence relations concerns determining the structure of this partial ordering. For a long time, many questions about this structure remained open, and it was notoriously difficult to obtain non-reducibility results. More recently, however, some progress has been made in establishing benchmarks within the $\leq_B$-hierarchy. In particular, an important breakthrough occurred in 2000 when Adams-Kechris [2] proved that the partial ordering of Borel subsets of $2^{\mathbb{N}}$ under inclusion embeds into the $\leq_B$ ordering on the subclass of *countable* Borel equivalence relations, which we shall define shortly.

As a first step towards describing the $\leq_B$-hierarchy, we introduce the so-called *smooth* and *hyperfinite* Borel equivalence relations. Writing $\mathrm{id}_{\mathbb{R}}$ for the identity relation on $\mathbb{R}$, the following result is a special case of a more general result of Silver [30] concerning co-analytic equivalence relations.

Theorem 1.2.2 (Silver)**.** *If E is a Borel equivalence relation with uncountably many classes, then $id_{\mathbb{R}} \leq_B E$.*

Hence $\mathrm{id}_{\mathbb{R}}$ – and any Borel equivalence relation bireducible with it – is a $\leq_B$-minimal element in the partial ordering of Borel equivalence relations with uncountably many classes.

Definition 1.2.3. *The Borel equivalence relation E is* smooth *iff $E \leq_B id_Z$ for some (equivalently every) uncountable standard Borel space Z.*

For example, it is easily checked that if the Borel equivalence relation E on the standard Borel space X admits a Borel tranversal, then E is smooth. (Here a Borel transversal is a

Borel subset $T \subseteq X$ which intersects every E-class in a single point.) While the converse does not hold for arbitrary Borel equivalence relations, we will later see that a countable Borel equivalence relation E is smooth iff E admits a Borel transversal.

The isomorphism relation on the space of countable divisible abelian groups is an example of a smooth equivalence relation. Similarly, if $\equiv$ is the equivalence relation defined on the space $\mathcal{G}$ of finitely generated groups by $G \equiv H$ iff $Th(G) = Th(H)$, then $\equiv$ is also smooth. For an example of a *non*-smooth Borel equivalence relation, we turn to the following:

Definition 1.2.4. *E_0 is the Borel equivalence relation defined on $2^{\mathbb{N}}$ by xE_0y iff $x(n) = y(n)$ for all but finitely many n.*

To see that E_0 is not smooth, suppose $f : 2^{\mathbb{N}} \to [0, 1]$ is a Borel reduction from E_0 to $\mathrm{id}_{[0,1]}$ and let μ be the usual product probability measure on $2^{\mathbb{N}}$. Then $f^{-1}([0, \frac{1}{2}])$ and $f^{-1}([\frac{1}{2}, 1])$ are Borel tail events, so by Kolmogorov's zero-one law, either $\mu(f^{-1}([0, \frac{1}{2}])) = 1$ or $\mu(f^{-1}([\frac{1}{2}, 1])) = 1$. Continuing to cut intervals in half in this manner, we obtain that f is μ-a.e. constant, a contradiction.

1.3. **Countable Borel Equivalence Relations.**

An important subclass of Borel equivalence relations consists of those with countable equivalence classes.

Definition 1.3.1. *A Borel equivalence relation on a standard Borel space is called* countable *if each of its equivalence classes is countable.*

The importance of this subclass stems in large part from the fact that each such equivalence relation can be realized as the orbit equivalence relation of a Borel action of a countable group. Of course, if G is a countable group and X a standard Borel G-space, then the corresponding orbit equivalence relation E_G^X is a countable Borel equivalence relation. But by a remarkable result of Feldman-Moore [10], the converse is also true:

Theorem 1.3.2 (Feldman-Moore)**.** *If E is a countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.*

Sketch of Proof. (For more details, see Srivastava [32, 5.8.13]). Let E be a countable Borel equivalence relation on the standard Borel space X. Since $E \subseteq X^2$ has countable sections, the Lusin-Novikov Uniformization Theorem [21, 18.10] implies that we can write E as a countable union of graphs of injective partial Borel functions, f_n : dom $f_n \to X$. Each f_n is easily modified into a Borel bijection $g_n : X \to X$ with the same "orbits." But then E is simply the orbit equivalence relation arising from the resulting Borel action of the group G generated by $\{ g_n \mid n \in \mathbb{N} \}$. □

Remark 1.3.3. The Lusin-Novikov Uniformization Theorem also implies that if E is a smooth countable Borel equivalence relation on the standard Borel space X, then E admits a Borel transversal. To see this, notice that if $f : X \to \mathbb{R}$ is a Borel reduction from E to $\mathrm{id}_{\mathbb{R}}$, then f is countable-to-one. Applying the Lusin-Novikov Uniformization Theorem to the Borel relation $R = \{ (f(x), x) \mid x \in X \}$, it follows that $f(X)$ is Borel and that there exists an injective Borel function $g : f(X) \to X$ such that $f(g(y)) = y$ for all $y \in f(X)$. Hence $T = g(f(X))$ is a Borel transversal for E.

Unfortunately, the countable group action given by the Feldman-Moore theorem is by no means canonical. For example, let us define the *Turing equivalence relation* $\equiv_T$ on $\mathcal{P}(\mathbb{N})$ by

$$A \equiv_T B \quad \text{iff} \quad A \leq_T B \text{ and } B \leq_T A,$$

where $\leq_T$ denotes Turing reducibility. Then $\equiv_T$ is clearly a countable Borel equivalence relation; and hence by the Feldman-Moore theorem, it must arise as the orbit equivalence relation induced by a Borel action of some countable group G on $\mathcal{P}(\mathbb{N})$. However, the proof of the Feldman-Moore theorem gives us no information about either the group G or its action, and so it is reasonable to ask:

Vague Question 1.3.4. *Can $\equiv_T$ be realized as the orbit equivalence relation of a "nice" Borel action of some countable group?*

We have earlier seen that there is a $\leq_B$-minimal Borel equivalence relation on an uncountable standard Borel space. On the other hand, by Friedman-Stanley [11], there does not exist a maximal relation in the setting of arbitrary Borel equivalence relations. However, by Dougherty-Jackson-Kechris [7], the subclass of countable Borel equivalence relations does admit a universal element.

Definition 1.3.5. *A countable Borel equivalence relation E is* universal *iff $F \leq_B E$ for every countable Borel equivalence relation F.*

This universal countable Borel equivalence relation can be realized as follows. Let $\mathbb{F}_\omega$ be the free group on infinitely many generators and define a Borel action of $\mathbb{F}_\omega$ on

$$(2^{\mathbb{N}})^{\mathbb{F}_\omega} = \{p \mid p : \mathbb{F}_\omega \to 2^{\mathbb{N}}\}$$

by setting

$$(g \cdot p)(h) = p(g^{-1}h), \quad p \in (2^{\mathbb{N}})^{\mathbb{F}_\omega}.$$

Let E_ω be the resulting orbit equivalence relation.

Claim 1.3.6. *E_ω is a universal countable Borel equivalence relation.*

Proof. Let X be a standard Borel space and let E be any countable Borel equivalence relation on X. Since every countable group is a homomorphic image of $\mathbb{F}_\omega$, the Feldman-Moore theorem implies that E is the orbit equivalence relation of a Borel action of $\mathbb{F}_\omega$. Let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of Borel subsets of X which separates points and define $f : X \to (2^{\mathbb{N}})^{\mathbb{F}_\omega}$ by $x \mapsto f_x$, where

$$f_x(h)(i) = 1 \text{ iff } x \in h(U_i).$$

Then f is injective and

$$\begin{aligned}(g \cdot f_x)(h)(i) = 1 \quad &\text{iff} \quad f_x(g^{-1}h)(i) = 1 \\ &\text{iff} \quad x \in g^{-1}h(U_i) \\ &\text{iff} \quad g \cdot x \in h(U_i) \\ &\text{iff} \quad f_{g\cdot x}(h)(i) = 1\end{aligned}$$

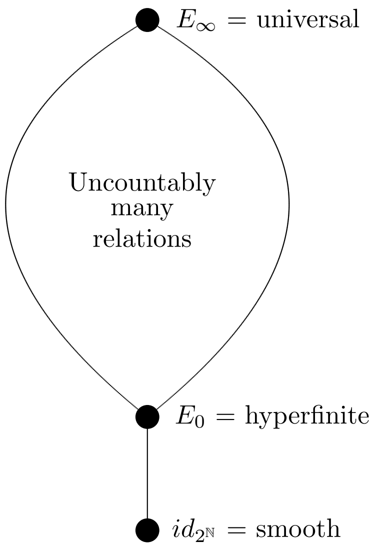
$\square$

Dougherty-Jackson-Kechris [7] have also shown that the orbit equivalence relation E_∞ arising from the translation action of the free group $\mathbb{F}_2$ on its powerset is a universal countable Borel equivalence relation. (Of course, any two universal countable Borel equivalence relations are Borel bireducible.)

We have now seen that within the class of countable Borel equivalence relations, there exist $\leq_B$-least and $\leq_B$-greatest such relations, up to $\sim_B$, with realizations given by $\mathrm{id}_{\mathbb{R}}$ and E_∞, respectively. It turns out that the minimal $\mathrm{id}_{\mathbb{R}}$ has an immediate $\leq_B$-successor:

Theorem 1.3.7 (Harrington-Kechris-Louveau [14]). *If E is a nonsmooth Borel equivalence relation, then $E_0 \leq_B E$.*

A Borel equivalence relation E is said to be *hyperfinite* if it can be written as an increasing union $E = \cup_n F_n$ of a sequence of finite Borel equivalence relations. (Here a Borel equivalence relation F is said to be *finite* if every F-class is finite.) It is easily shown that E_0 is hyperfinite; and in fact, every nonsmooth hyperfinite countable Borel equivalence relation is Borel bireducible with E_0. Furthermore, by a result of Dougherty-Jackson-Kechris [7], if E is a countable Borel equivalence relation, then E can be realized as the orbit equivalence relation of a Borel $\mathbb{Z}$-action if and only if $E \leq_B E_0$. (There will be a further discussion of the class of hyperfinite equivalence relations in Subsection 4.5.1.) Finally, by the previously mentioned result of Adams-Kechris [2], we know that there exist $2^{\aleph_0}$ distinct countable Borel equivalence relations up to Borel bireducibility. Combining these basic facts gives the following picture of the universe of countable Borel equivalence relations.



Given this picture, one can ask where a particular countable Borel equivalence relation lies relative to the known benchmarks. In the following section, we shall consider this question for the Turing equivalence relation $\equiv_T$. Here it is interesting to note that Martin has conjectured that $\equiv_T$ is *not* universal, while Kechris has conjectured that it is. However, despite some progress, which we will discuss below, this important problem remains open.

1.4. **Turing Equivalence and The Martin Conjectures.** We first define the set of *Turing degrees* to be the collection

$$\mathcal{D} = \{\, \mathbf{a} = [A]_{\equiv_T} \mid A \in \mathcal{P}(\mathbb{N})\}$$

of $\equiv_T$-classes. A subset $X \subseteq \mathcal{D}$ is said to be *Borel* iff $X^* = \bigcup\{\mathbf{a} \mid \mathbf{a} \in X\}$ is a Borel subset of $\mathcal{P}(\mathbb{N})$. It is well-known that if E is a countable Borel equivalence relation on a standard Borel space X, then the quotient Borel space $X/E = \{\, [x]_E \mid x \in X \,\}$ is a standard Borel space if and only if E is smooth. (If X/E is a standard Borel space, then the map $x \mapsto [x]_E$ witnesses that E is smooth. Conversely, if E is smooth, then E admits a Borel transversal T and X/E is isomorphic to the standard Borel space T.) In particular, since $\equiv_T$ is not smooth, it follows that $\mathcal{D}$ is *not* a standard Borel space.

For $\mathbf{a}, \mathbf{b} \in \mathcal{D}$, we define $\mathbf{a} \leq \mathbf{b}$ iff $A \leq_T B$ for each $A \in \mathbf{a}$ and $B \in \mathbf{b}$; and for each $\mathbf{a} \in \mathcal{D}$, we define the corresponding *cone* to be $C_{\mathbf{a}} = \{\mathbf{b} \in \mathcal{D} \mid \mathbf{a} \leq \mathbf{b}\}$. Of course, each cone $C_{\mathbf{a}}$ is a Borel subset of $\mathcal{D}$.

Theorem 1.4.1 (Martin)**.** *If $X \subseteq \mathcal{D}$ is Borel, then for some $\mathbf{a} \in \mathcal{D}$, either $C_{\mathbf{a}} \subseteq X$ or $C_{\mathbf{a}} \subseteq \mathcal{D} \setminus X$.*

Proof. Let $X \subseteq \mathcal{D}$ be Borel and consider the 2-player game $G(X^*)$

$$a = a(0)a(1)a(2)\cdots, \quad \text{where each } a(n) \in 2,$$

such that Player 1 wins iff $a \in X^*$. Then $G(X^*)$ is Borel and hence is determined. Suppose, for example, that $\varphi : 2^{<\mathbb{N}} \to 2$ is a winning strategy for Player 1. We claim that $C_\varphi \subseteq X$.

To see this, suppose that $\varphi \leq_T x$ and let Player 2 play $x = a(1)a(3)a(5)\cdots$. Then $y = \varphi(x) \in X^*$ and $x \equiv_T y$. It follows that $x \in X^*$. $\square$

For later use, notice that if $X \subseteq \mathcal{D}$ is Borel, then X contains a cone iff X is $\leq_T$-cofinal in the set $\mathcal{D}$ of Turing degrees.

In a similar fashion, we define a function $f : \mathcal{D} \to \mathcal{D}$ to be *Borel* iff there exists a Borel function $\varphi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $f([A]_{\equiv_T}) = [\varphi(A)]_{\equiv_T}$. We are now ready to state the following conjecture of Martin, which (as we will soon see) implies that $\equiv_T$ is not universal.

Conjecture 1.4.2 (Martin)**.** *If $f : \mathcal{D} \to \mathcal{D}$ is Borel, then either f is constant on a cone or else $f(\mathbf{a}) \geq \mathbf{a}$ on a cone.*

While this conjecture remains open, there do exist some partial results of Slaman-Steel [31] that point in its direction:

Theorem 1.4.3 (Slaman-Steel)**.** *If $f : \mathcal{D} \to \mathcal{D}$ is Borel and $f(\mathbf{a}) < \mathbf{a}$ on a cone, then f is constant on a cone.*

Theorem 1.4.4 (Slaman-Steel)**.** *If the Borel map $f : \mathcal{D} \to \mathcal{D}$ is uniformly invariant, then either f is constant on a cone or else $f(\mathbf{a}) \geq \mathbf{a}$ on a cone.*

(The definition of a uniformly invariant map can be found in Slaman-Steel [31].) Next, following Dougherty-Kechris [8], we will show that the Martin conjecture implies that $\equiv_T$ is not universal. First recall that, by Dougherty-Jackson-Kechris [7], if E, F are countable Borel equivalence relations on the standard Borel spaces X, Y respectively, then $E \sim_B F$ iff there exist Borel complete sections $A \subseteq X$, $B \subseteq Y$ such that $E \restriction A \cong F \restriction B$ via a Borel isomorphism. (Here a Borel subset $A \subseteq X$ is said to be a *complete section* if A intersects every E-class.) In particular, if $\equiv_T$ is universal, then $(\equiv_T \times \equiv_T) \sim_B \equiv_T$; and hence there exist Borel complete sections $Y \subseteq \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ and $Z \subseteq \mathcal{P}(\mathbb{N})$ such that $(\equiv_T \times \equiv_T) \restriction Y \cong \equiv_T \restriction Z$ via a Borel isomorphism φ. Let $f : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ be the Borel pairing function induced by φ. Then fixing $\mathbf{d}_0 \neq \mathbf{d}_1 \in \mathcal{D}$, we can define Borel maps $f_i : \mathcal{D} \to \mathcal{D}$ by $f_i(\mathbf{a}) = f(\mathbf{d}_i, \mathbf{a})$. By the Martin Conjecture, $f_i(\mathbf{a}) \geq \mathbf{a}$ on a cone and so each ran f_i is a cofinal Borel subset of $\mathcal{D}$. But this means that each ran f_i contains a cone, which is impossible since ran $f_0 \cap$ ran $f_1 = \emptyset$.

In constrast, let $\equiv_A$ be the *arithmetic equivalence relation* defined on $\mathcal{P}(\mathbb{N})$ by

$$B \equiv_A C \quad \text{iff} \quad B \leq_A C \text{ and } C \leq_A B,$$

where $\leq A$ denotes arithmetic reducibility. Then Slaman-Steel have shown that $\equiv_A$ is a universal countable Borel equivalence relation. One might take this as evidence that $\equiv_T$ is also universal. However, as Slaman has pointed out, an important difference between the two cases is that the arithmetic degrees have less closure with respect to arithmetic equivalences than the Turing degrees do with respect to recursive equivalences.

2. Second Session

2.1. **The Fundamental Question in the Theory of Countable Borel Equivalence Relations.** We have already seen that, by the Feldman-Moore theorem, every countable Borel equivalence relation on a standard Borel space arises as the orbit equivalence relation of a Borel action of a suitable countable group. However, we have also seen that this action is not canonically determined, and that it is sometimes difficult to express a given countable Borel equivalence relation as the orbit equivalence relation arising from a "natural" group action. Since many of the techniques currently available for analyzing countable Borel equivalence relations involve properties of the groups and actions from which they arise, one of the fundamental questions in the theory concerns the extent to which an orbit equivalence relation

E_G^X determines the group G and its action on X. Ideally one would hope for the complexity of E_G^X to reflect the complexity of G, so that relations E_G^X and E_H^X can be distinguished (in the sense of $\leq_B$) by distinguishing G from H.

Of course, strong hypotheses on a countably infinite group G and its action on a standard Borel space X must be made if there is to be any hope of recovering G and its action from E_G^X. For example, let G be any countable group and consider the Borel action of G on $G \times [0,1]$ defined by $g \cdot (h,r) = (gh,r)$. Then the Borel map $(h,r) \mapsto (1_G, r)$ selects a point in each G-orbit, and so the corresponding orbit equivalence relation is smooth. Notice, however, that this action does not admit an invariant probability measure. In fact, we have the following simple but important observation.

Definition 2.1.1. *A Borel action of a countable group G on a standard Borel space X is said to be* free *iff $g \cdot x \neq x$ for all $1 \neq g \in G$ and $x \in X$. In this case, we say that X is a* free standard Borel G-space.

Proposition 2.1.2. *If a countably infinite group G acts freely on X and preserves a probability measure μ, then E_G^X is not smooth.*

Proof. If E is smooth, then E admits a Borel transversal $T \subseteq X$. But since G acts freely, it follows that X can be expressed as the disjoint union $X = \bigsqcup_{g \in G} g(T)$, which means that T is not μ-measurable. □

The following two theorems show that if we are serious about recovering the group G and its action from E_G^X, then it is necessary to assume that G satisfies *both* of the hypotheses of Proposition 2.1.2

Theorem 2.1.3 (Dougherty-Jackson-Kechris [7])**.** *Let G be a countable group and let X be a standard Borel G-space. If X does not admit a G-invariant probability measure, then for every countable group $H \supset G$, there exists a Borel action of H on X such that $E_H^X = E_G^X$. Furthermore, if G acts freely on X, then there exists a free Borel action of H on X such that $E_H^X = E_G^X$.*

In order to see that it is also necessary to assume that the action of G on X is free, consider the associated homomorphism $\pi : G \to \mathrm{Sym}(X)$. Of course, if $\ker \pi \neq 1$, then we cannot recover G from its action on X. Thus it is certainly necessary to assume that G acts faithfully on X. Following Miller [27], the action of G on X is said to be *everywhere faithful* if G acts faithfully on every G-orbit. The following is an easy consequence of a much more general result of Miller [27].

Theorem 2.1.4 (Miller). *Suppose that E is a countable Borel equivalence relation such that every E-class is infinite. Then there exists an uncountable family $\mathcal{F}$ of pairwise non-embeddable countable groups such that E can be realized as the orbit equivalence relation of an everywhere faithful Borel action of G for every $G \in \mathcal{F}$.*

Definition 2.1.5. *A countable Borel equivalence relation in which every E-class is infinite is called* aperiodic.

Consequently, we shall be especially concerned with *free*, *measure-preserving* Borel actions of countable groups on standard Borel probability spaces. A natural question, then, is whether we can *always* hope for this setting:

Question 2.1.6. *Let E be a nonsmooth countable Borel equivalence relation. Does there necessarily exist a countable group G with a free measure-preserving Borel action on a standard probability space (X, μ) such that $E \sim_B E_G^X$?*

We first observe that half of this question is easily answered: namely, if E is a countable Borel equivalence relation on an uncountable standard Borel space Y, then there exists a countable group G and a standard Borel G-space X such that G preserves a nonatomic probability measure μ on X, and $E \sim_B E_G^X$. To see this, let G be a countable group with a Borel action on Y such that $E_G^Y = E$. Then we can regard $X = Y \sqcup [0,1]$ as a standard Borel G-space by letting G act trivially on $[0,1]$. If we regard the usual probability measure μ on $[0,1]$ as a probability measure on X which concentrates on $[0,1]$, then E_G^X satisfies our requirements. At this point, it is convenient to introduce two more definitions.

Definition 2.1.7. *The countable Borel equivalence relation E on X is* free *iff there exists a countable group G with a free Borel action on X such that $E_G^X = E$.*

Definition 2.1.8. *The countable Borel equivalence relation E is* essentially free *iff there exists a free countable Borel equivalence relation F such that $E \sim_B F$.*

In view of the above discussion, it is clear that we should replace Question 2.1.6 by the following question (which no longer mentions an invariant measure).

Question 2.1.9 (Jackson-Kechris-Louveau [19]). *Is every countable Borel equivalence relation essentially free?*

2.2. **Essentially Free Countable Borel Equivalence Relations.** Before answering Question 2.1.9, it will be helpful to first list some closure properties of essential freeness, which we will state without proof.

Theorem 2.2.1 (Jackson-Kechris-Louveau [19]). *Let E, F be countable Borel equivalence relations on the standard Borel spaces X, Y respectively.*

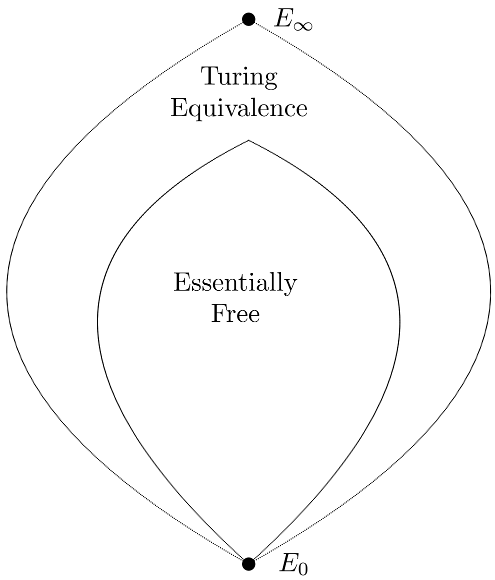
- *If $E \leq_B F$ and F is essentially free, then so is E.*
- *If $E \subseteq F$ and F is essentially free, then so is E.*

It follows that *every* countable Borel equivalence relation is essentially free iff the universal countable Borel equivalence relation E_∞ is essentially free. The following result will be proved in Subsection 2.3.

Theorem 2.2.2 (Thomas 2006, [36]). *The class of essentially free countable Borel equivalence relations does not admit a universal element. In particular, E_∞ is not essentially free.*

Thus, unfortunately, the answer to Question 2.1.6 is no. As a corollary to 2.2.2 and 2.2.1, we observe that $\equiv_T$ is not essentially free; for identifying the free group $\mathbb{F}_2$ with a suitably chosen group of recursive permutations of $\mathbb{N}$, we have that $E_\infty \subseteq \equiv_T$.

This gives us the following map of the universe of nonsmooth countable Borel equivalence relations.



2.3. **Bernoulli Actions, Popa Superrigidity, and the Proof of Theorem 2.2.2.** In this section, we will state a striking consequence of Popa's Superrigidity Theorem, which easily implies Theorem 2.2.2. We will begin with a short discussion of *Bernoulli actions*.

By a Bernoulli action, we mean the shift action of a countably infinite discrete group G on its powerset $\mathcal{P}(G) = 2^G$, defined by $g \cdot x(h) = x(g^{-1}h)$. (This is a special case of the notion as it appears in Popa [29]). Under this action, the usual product probability measure μ on 2^G is G-invariant and the *free part*

$$\mathcal{P}^*(G) = (2)^G = \{x \in 2^G \mid g \cdot x \neq x \text{ for all } 1 \neq g \in G\}$$

has μ-measure 1. We let E_G denote the corresponding orbit equivalence relation on $(2)^G$ and make the following observation:

Proposition 2.3.1. *If $G \leq H$, then $E_G \leq_B E_H$.*

Proof. The inclusion map $\mathcal{P}^*(G) \hookrightarrow \mathcal{P}^*(H)$ is a Borel reduction from E_G to E_H. $\square$

Now we just need a few more preliminary definitions before we can state the consequence of Popa's theorem which we will need to prove Theorem 2.2.2.

Definition 2.3.2. *If E, F are Borel equivalence relations on the standard Borel spaces X, Y respectively, then a Borel map $f : X \to Y$ is said to be a* homomorphism *from E to F if*

$$x \, E \, y \quad \Longrightarrow \quad f(x) \, F \, f(y)$$

for all x, $y \in X$.

Definition 2.3.3. *If μ is an E-invariant probability measure on X, then the Borel homomorphism $f : X \to Y$ from E to F is said to be μ-*trivial *if there exists a Borel subset $Z \subseteq X$ with $\mu(Z) = 1$ such that f maps Z into a single F-class.*

Definition 2.3.4. *If G and H are countable groups, then the homomorphism $\pi : G \to H$ is a* virtual embedding *if $|ker\ \pi| < \infty$.*

Now we are finally ready to state the consequence of Popa's Cocycle Superrigidity Theorem [29] that we shall use to prove Theorem 2.2.2. We shall discuss Popa's theorem and deduce the following consequence at a later point in these notes.

Theorem 2.3.5. *Let $G = SL_3(\mathbb{Z}) \times S$, where S is any countable group. Let H be any countable group and let Y be a free standard Borel H-space. If there exists a μ-nontrivial Borel homomorphism from E_G to E_H^Y, then there exists a virtual embedding $\pi : G \to H$.*

We observe that, in particular, this conclusion holds if there exists a Borel subset $Z \subseteq (2)^G$ with $\mu(Z) = 1$ such that $E_G \restriction Z \leq_B E_H^Y$. Theorem 2.2.2 is then an immediate corollary of the following:

Theorem 2.3.6. *If E is an essentially free countable Borel equivalence relation, then there exists a countable group G such that $E_G \nleq_B E$.*

Proof. We can suppose that $E = E_H^X$ is realized by a free Borel action on X of the countable group H. Let L be a finitely generated group which does not embed into H. Let $S = L * \mathbb{Z}$ and let $G = SL_3(\mathbb{Z}) \times S$. Then G has no finite normal subgroups and so there does not exist a virtual embedding $\pi : G \to H$. It follows that $E_G \nleq_B E_H^X$. □

2.4. **Free and Non-Essentially Free Countable Borel Equivalence Relations.** We will now use 2.3.5 to show that there are continuum many free countable Borel equivalence relations. For each prime $p \in \mathbb{P}$, let $A_p = \bigoplus_{i=0}^{\infty} C_p$, where C_p is the cyclic group of order p ; and for each subset $C \subseteq \mathbb{P}$, let

$$G_C = SL_3(\mathbb{Z}) \times \bigoplus_{p \in C} A_p.$$

Then the desired result is an immediate consequence of the following:

Theorem 2.4.1. *If $C, D \subseteq \mathbb{P}$, then $E_{G_C} \leq_B E_{G_D}$ iff $C \subseteq D$.*

Proof. If $C \subseteq D$, then $G_C \leq G_D$, and hence $E_{G_C} \leq_B E_{G_D}$. Conversely, applying 2.3.5, if $E_{G_C} \leq_B E_{G_D}$, then there exists a virtual embedding $\pi : G_C \to G_D$. Since $SL_3(\mathbb{Z})$ contains a torsion-free subgroup of finite index, it follows that for each $p \in C$, the cyclic group C_p embeds into $\bigoplus_{q \in D} A_q$ and this implies that $p \in D$. □

We will now show that there also exist continuum many *non*-essentially free countable Borel equivalence relations. We begin by introducing the notion of *ergodicity*.

Definition 2.4.2. *Let G be a countable group and let X be a standard Borel G-space with G-invariant probability measure μ. Then the action of G on (X, μ) is said to be* ergodic *if $\mu(A) = 0$ or $\mu(A) = 1$ for every G-invariant Borel subset $A \subseteq X$.*

For example, every countable group G acts ergodically on $((2)^G, \mu)$. (This is a consequence of Theorem 3.1.3.) The following characterization of ergodicity is well-known.

Theorem 2.4.3. *If μ is a G-invariant probability measure on the standard Borel G-space X, then the following statements are equivalent.*

- *The action of G on (X, μ) is ergodic.*
- *If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction M$ is a constant function.*

Finally we need just one more definition.

Definition 2.4.4. *The countable groups G, H are said to be* virtually isomorphic *if there exist finite normal subgroups $N \lhd G$, $M \lhd H$ such that $G/N \cong H/M$.*

The groups given by the following lemma will be used below to construct the desired examples of non-essentially free countable Borel equivalence relations. (The proof of Lemma 2.4.5 can be found in Thomas [36].)

Lemma 2.4.5. *There exists a Borel family $\{S_x \mid x \in 2^{\mathbb{N}}\}$ of finitely generated groups such that if $G_x = SL_3(\mathbb{Z}) \times S_x$, then the following conditions hold:*

- *If $x \neq y$, then G_x and G_y are not virtually isomorphic.*
- *If $x \neq y$, then G_x does not virtually embed in G_y.*

Now, for each Borel subset $A \subseteq 2^{\mathbb{N}}$, let $E_A = \bigsqcup_{x \in A} E_{G_x}$ be the corresponding *smooth disjoint union*; i.e. E_A is the countable Borel equivalence relation defined on the standard Borel space

$$\bigsqcup_{x \in A} (2)^{G_x} = \{\, (x, r) \mid x \in A, r \in (2)^{G_x} \,\}$$

defined by

$$(x, r)\; E_A\; (y, s) \quad \iff \quad x = y \text{ and } r\; E_{G_x}\; s.$$

Lemma 2.4.6. *If the Borel subset $A \subseteq 2^{\mathbb{N}}$ is uncountable, then E_A is not essentially free.*

Proof. Suppose that $E_A \leq_B E_H^Y$, where H is a countable group and Y is a free standard Borel H-space. Then for each $x \in A$, we have that $E_{G_x} \leq_B E_H^Y$ and so there exists a virtual embedding $\pi_x : G_x \to H$. Since A is uncountable and each G_x is finitely generated, there exist $x \neq y \in A$ such that $\pi_x[G_x] = \pi_y[G_y]$. But then G_x, G_y are virtually isomorphic, which is a contradiction. □

Lemma 2.4.7. *$E_A \leq_B E_B$ iff $A \subseteq B$.*

Proof. It is clear that if $A \subseteq B$, then $E_A \leq_B E_B$. Conversely, suppose that $E_A \leq_B E_B$ and that $A \not\subseteq B$. Let $x \in A \setminus B$. Then there exists a Borel reduction

$$f : (2)^{G_x} \to \bigsqcup_{y \in B} (2)^{G_y}$$

from E_{G_x} to E_B. By ergodicity, there exists a μ_x-measure 1 subset of $(2)^{G_x}$ which maps to a fixed $(2)^{G_y}$. This yields a μ_x-nontrivial Borel homomorphism from E_{G_x} to E_{G_y} and so G_x virtually embeds into G_y, which is a contradiction. □

Of course, the existence of uncountably many non-essentially free countable Borel equivalence relations is an immediate consequence of Lemmas 2.4.6 and 2.4.7.

3. Third Session

3.1. **Ergodicity, Strong Mixing, and Borel Cocycles.** In this section, we will discuss some of the background material which is necessary in order to understand the statement of Popa's Cocycle Superrigidity Theorem and the proof of Theorem 2.3.5. As usual, if a countable group G acts on a standard probability space (X, μ), then we assume that the action is both free and measure-preserving, so that we may stand some chance of recovering the group G and its action on X from the orbit equivalence relation E_G^X.

Recall now that a measure-preserving action of a countable group G on a standard Borel probability G-space (X, μ) is *ergodic* iff every G-invariant Borel subset of X is null or conull; equivalently, the action of G on (X, μ) is ergodic iff whenever Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \upharpoonright M$ is a constant function. In particular, ergodicity is a natural obstruction to smoothness: if G acts ergodically on the standard Borel probability G-space (X, μ), then the corresponding orbit equivalence relation E_G^X is not smooth.

Definition 3.1.1. *The action of G on the standard probability space (X, μ) is* strongly mixing *if for any Borel subsets $A, B \subseteq X$, we have that*

$$\mu(g(A) \cap B) \to \mu(A) \cdot \mu(B) \quad \textit{as } g \to \infty.$$

In other words, if $\langle g_n \mid n \in \mathbb{N} \rangle$ *is* any *sequence of distinct elements of* G*, then*

$$\lim_{n\to\infty} \mu(g_n(A) \cap B) = \mu(A) \cdot \mu(B).$$

Mixing is a strong form of ergodicity. Indeed, suppose that the action of G on (X, μ) is strongly mixing and let $A \subseteq X$ be a G-invariant Borel subset. Then

$$\mu(A)^2 = \lim_{g\to\infty} \mu(g(A) \cap A) = \lim_{g\to\infty} \mu(A) = \mu(A),$$

which implies that $\mu(A) = 0$ or 1. Hence strongly mixing actions are ergodic. However, unlike ergodicity, strong mixing is a property that passes to infinite subgroups.

Observation 3.1.2. *If the action of* G *on* (X, μ) *is strongly mixing and* $H \leq G$ *is an infinite subgroup of* G*, then the action of* H *on* (X, μ) *is also strongly mixing.*

That the above observations actually apply to our setting is given by the following:

Theorem 3.1.3. *The action of* G *on* $((2)^G, \mu)$ *is strongly mixing.*

Sketch Proof. Consider the special case when there exist finite subsets $S, T \subseteq G$ and subsets $\mathcal{F} \subseteq 2^S$, $\mathcal{G} \subseteq 2^T$ such that $A = \{f \in (2)^G \mid f \restriction S \in \mathcal{F}\}$ and $B = \{f \in (2)^G \mid f \restriction T \in \mathcal{G}\}$. (Of course, the "cylinder" sets of this form generate the measure μ.) If $\langle g_n \mid n \in \mathbb{N} \rangle$ is a sequence of distinct elements of G, then $g_n(S) \cap T = \emptyset$ for all but finitely many n. This means that $g_n(A)$ and B are independent events and so

$$\mu(g_n(A) \cap B) = \mu(g_n(A)) \cdot \mu(B) = \mu(A) \cdot \mu(B).$$

It follows that $\lim_{n\to\infty} \mu(g_n(A) \cap B) = \mu(A) \cdot \mu(B)$. □

The final important concept which we must introduce before stating Popa's Theorem is that of a Borel cocycle. Let G, H be countable discrete groups and let X be a standard Borel G-space with invariant Borel probability measure μ. Then a Borel map $\alpha : G \times X \to H$ is a *cocycle* iff α satisfies the cocycle identity

$$\forall g, h \in G \quad \alpha(hg, x) = \alpha(h, gx)\, \alpha(g, x) \quad \mu\text{-a.e.}(x).$$

If $\beta : G \times X \to H$ is another cocycle into H, then we say that α and β are *equivalent*, and write $\alpha \sim \beta$, iff there is a Borel map $b : X \to H$ such that

$$\forall g \in G \quad \beta(g, x) = b(gx)\, \alpha(g, x)\, b(x)^{-1} \quad \mu\text{-a.e.}(x).$$

It is clear that $\sim$ is an equivalence relation on the set of cocycles $G \times X \to H$.

In these lectures, cocycles $\alpha : G \times X \to H$ will always arise from Borel homomorphisms into free standard Borel H-spaces in the following way. Suppose that Y is a free standard Borel H-space and that f is a Borel homomorphism from E_G^X to E_H^Y. Then we can define a corresponding cocycle $\alpha : G \times X \to H$ by

$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

Moreover, if α is the cocycle corresponding in this manner to the Borel homomorphism $f : X \to Y$ and if $b : X \to H$ is any Borel function, then the map $f' : X \to Y$ defined by $f'(x) = b(x)f(x)$ is also a Borel homomorphism, and the corresponding cocycle β is equivalent to α via the the equation

$$\beta(g, x) = b(gx)\,\alpha(g, x)\,b(x)^{-1}.$$

Equivalence of cocycles can be easily visualized with the aid of the following diagram:

$$\begin{array}{ccc} G & & H \\ \boxed{\begin{array}{c} x \\ {\scriptstyle g}\Big\downarrow \\ g \cdot x \end{array}} & \xrightarrow{f} & \boxed{\begin{tikzcd} f(x) \arrow[r, "b(x)"] \arrow[d, "\alpha(g,x)"'] & f'(x) \arrow[d, "\beta(g,x)"] \\ f(g \cdot x) \arrow[r, "b(g\cdot x)"] & f'(g \cdot x) \end{tikzcd}} \\ (X, \mu) & & Y \end{array}$$

Notice that if the cocycle $\alpha : G \times X \to H$ is actually a function of only one variable, i.e. the value of $\alpha(g, x) = \alpha(g)$ is independent of x, then α is a group homomorphism from G to H; and if $f : X \to Y$ is the corresponding Borel homomorphism, then $(G, X) \xrightarrow{(\alpha,f)} (H, Y)$ is a permutation group homomorphism.

3.2. **Popa's Cocycle Superrigidity Theorem and the Proof of Theorem 2.3.5.** We are almost ready to state Popa's Cocycle Superrigidity Theorem [29]. But first we need to present a short discussion concerning the notions of amenable, nonamenable and Kazhdan groups.

A countable (discrete) group G is *amenable* if there exists a finitely additive G-invariant probability measure $\nu : \mathcal{P}(G) \to [0, 1]$ defined on every subset of G. For example, finite groups are amenable and abelian groups are amenable. Furthermore, the class of amenable is closed under taking subgroups, forming extensions and taking direct limits. In particular, solvable groups are also amenable. On the other hand, nonabelian free groups are nonamenable; and for many years, it was a open problem whether every countable nonamenable group contained a nonabelian free subgroup, until Ol'shanskii [28] constructed a periodic nonamenable group in 1980. (An excellent introduction to the theory of amenable groups can be found in Wagon [37].)

In many senses, the opposite of the notion of an amenable group is that of a Kazhdan group. For our purposes in these notes, it is enough to know that if $m \geq 3$, then $SL_m(\mathbb{Z})$ is a Kazhdan group. However, for the sake of completeness, we will provide the formal definition. So let G be a countably infinite group and let $\pi : G \to U(\mathcal{H})$ be a unitary representation of G on the separable Hilbert space $\mathcal{H}$. Then π *almost admits invariant vectors* if for every $\varepsilon > 0$ and every finite subset $K \subseteq G$, there exists a unit vector $v \in \mathcal{H}$ such that $||\pi(g).v - v|| < \varepsilon$ for all $g \in K$. We say that G is a *Kazhdan group* if for every unitary representation π of G, if π almost admits invariant vectors, then π has a non-zero invariant vector. (An excellent introduction to the theory of Kazhdan groups can be found in Lubotzky [24].)

We are finally ready to state (a special case of) Popa's Cocycle Superrigidity Theorem [29].

Theorem 3.2.1 (Popa)**.** *Let Γ be a countably infinite Kazhdan group and let G be a countable group such that $\Gamma \lhd G$. If H is any countable group, then every Borel cocycle*

$$\alpha : G \times (2)^G \to H$$

is equivalent to a group homomorphism of G into H.

For example, we may let $\Gamma = SL_n(\mathbb{Z})$ for any $n \geq 3$ and $G = \Gamma \times S$, where S is any countable group. We are now ready to prove Theorem 2.3.5.

Proof of Theorem 2.3.5. Let $G = SL_3(\mathbb{Z}) \times S$ and let Y be a free standard Borel H-space, where S and H are any countable groups. Suppose the $f : (2)^G \to Y$ is a μ-nontrivial Borel homomorphism from E_G to E_H^Y, where E_G denotes the orbit equivalence relation of the Bernoulli action of G on $((2)^G, \mu)$. Then we can define a Borel cocycle $\alpha : G \times (2)^G \to H$ by

$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot f(x) = f(g \cdot x).$$

By Theorem 3.2.1, after deleting a null set of $(2)^G$ and adjusting f if necessary, we can suppose that $\alpha : G \to H$ is a group homomorphism.

Suppose that $K =$ ker α is infinite. Note that if $k \in K$, then $f(k \cdot x) = \alpha(k) \cdot x = f(x)$ and so $f : (2)^G \to X$ is K-invariant. Also since the action of G is strongly mixing, it follows that K acts ergodically on $((2)^G, \mu)$. But then the K-invariant function $f : (2)^G \to X$ is μ-a.e. constant, which is a contradiction. $\square$

3.3. **Torsion-free Abelian Groups of Finite Rank.** Recall that an additive subgroup $G \leq \mathbb{Q}^n$ has rank n iff G contains n linearly independent elements; and that we have previously defined the standard Borel space $R(\mathbb{Q}^n)$ of torsion-free abelian groups of rank n to be

$$R(\mathbb{Q}^n) = \{A \leq \mathbb{Q}^n \mid A \text{ contains a basis of } \mathbb{Q}^n\}.$$

Recall also that for $A, B \in R(\mathbb{Q}^n)$, we have that

$$A \cong B \quad \text{iff} \quad \text{there exists } g \in GL_n(\mathbb{Q}) \text{ such that } g(A) = B.$$

Thus the isomorphism relation $\cong_n$ on $R(\mathbb{Q}^n)$ is the orbit equivalence relation arising from the natural action of $GL_n(\mathbb{Q})$ on $R(\mathbb{Q}^n)$.

In 1937, Baer [4] gave a satisfactory classification of the rank 1 groups, which showed that $\cong_1$ is hyperfinite. In 1938, Kurosh [23] and Malcev [25] independently gave *unsatisfactory* classifications of the higher rank groups. In light of this failure to classify even the rank 2 groups in a satisfactory way, Hjorth-Kechris [17] conjectured in 1996 that the isomorphism relation for the torsion-free abelian groups of rank 2 was countable universal. As an initial step towards establishing this result, Hjorth [15] then proved in 1998 that the classification problem for the rank 2 groups is strictly harder than that for the rank 1 groups; that is, Hjorth proved that $\cong_1 <_B \cong_2$. Soon afterwards, making essential use of the techniques of Hjorth [15] and Adams-Kechris [2], Thomas obtained the following [34]:

Theorem 3.3.1 (Thomas 2000)**.** *The complexity of the classification problem for the torsion-free abelian groups of rank n increases strictly with the rank n.*

Of course, this implies that none of the relations $\cong_n$ is countable universal. It remained open, however, whether the isomorphism relation on the space of torsion-free abelian groups of *finite* rank was countable universal. In 2006 [36], making use of Popa's Cocycle Superrigidity Theorem, Thomas was finally able to show that it is not.

Theorem 3.3.2 (Thomas 2006)**.** *The isomorphism relation on the space of torsion-free abelian groups of finite rank is not countable universal.*

In the next two sections, we shall present an outline of the proof of Theorem 3.3.2. We will begin by introducing the notion of E_0-ergodicity, which will play an important role at the end of the proof.

3.4. **E_0-ergodicity.** The following is another useful strengthening of ergodicity.

Definition 3.4.1. *Let E, F be countable Borel equivalence relations on the standard Borel spaces X, Y and let μ be an E-invariant probability measure on X. Then E is said to be* F-ergodic *iff every Borel homomorphism $f : X \to Y$ from E to F is μ-trivial.*

Thus $\mathrm{id}_{\mathbb{R}}$-ergodicity coincides with the usual notion of ergodicity. Furthermore, observe that if E is F-ergodic and $F' \leq_B F$, then E is also F'-ergodic. The following characterization of E_0-ergodicity is due to Jones-Schmidt [20].

Definition 3.4.2. *Let $E = E_G^X$ be a countable Borel equivalence relation and let μ be an E-invariant probability measure on X. Then E has* nontrivial almost invariant subsets *iff there exists a sequence of Borel subsets $\langle A_n \subseteq X \mid n \in \mathbb{N} \rangle$ satisfying the following conditions:*

- $\mu(g \cdot A_n \triangle A_n) \to 0$ *for all $g \in G$.*
- *There exists $\delta > 0$ such that $\delta < \mu(A_n) < 1 - \delta$ for all $n \in \mathbb{N}$.*

Theorem 3.4.3 (Jones-Schmidt)**.** *Suppose that E is a countable Borel equivalence relation on the standard Borel space X and that μ is an ergodic E-invariant probability measure. Then E is E_0-ergodic iff E has no nontrivial almost invariant subsets.*

This can in turn be used to prove the following:

Theorem 3.4.4 (Jones-Schmidt)**.** *If G is a countable group and $H \leq G$ is a nonamenable subgroup, then the shift action of H on $((2)^G, \mu)$ is E_0-ergodic.*

Finally, we remark for later use that if E is E_0-ergodic and F is hyperfinite, then E is also F-ergodic. We are now ready to commence our sketch of the proof of the non-universality of the isomorphism relation on the space of torsion-free abelian groups of finite rank.

3.5. **The Non-universality of the Isomorphism Relation on Torsion-free Abelian Groups of Finite Rank.** Roughly speaking, the strategy of our proof will be as follows. The results of Jackson-Kechris-Louveau [19, Section 5.2] easily imply that a smooth disjoint union of *countably* many essentially free countable Borel equivalence relations is itself essentially free; and we already know that the class of essentially free countable Borel equivalence relations does not admit a universal element. Since the isomorphism relation on the space of torsion-free abelian groups of finite rank is the smooth disjoint union of the $\cong_n$ relations, $n \geq 1$, it would thus suffice to show that each $\cong_n$ is essentially free. Unfortunately, it appears to be difficult to determine whether this is true even for the case when $n = 2$. However, we shall show that the coarser quasi-isomorphism relation is "(hyperfinite)-by-(essentially free)", and this will turn out be enough. We will now proceed with the details.

Let $G = SL_3(\mathbb{Z}) \times S$, where S is a suitably chosen countable group that we shall specify at a later stage in the proof. Let $E = E_G$ be the orbit equivalence relation arising from the action of G on $((2)^G, \mu)$. Suppose that

$$f : (2)^G \to \bigsqcup_{n \geq 1} R(\mathbb{Q}^n)$$

is a Borel reduction from E to the isomorphism relation for the torsion-free abelian groups of finite rank. After deleting a null set of $(2)^G$ if necessary, we may assume that f takes values in $R(\mathbb{Q}^n)$ for some *fixed* $n \geq 1$.

At this point, we would like to define a Borel cocycle corresponding to f, but unfortunately $GL_n(\mathbb{Q})$ does not act freely on $R(\mathbb{Q}^n)$. In fact, the stabilizer of each $B \in R(\mathbb{Q}^n)$ under the action of $GL_n(\mathbb{Q})$ is precisely its automorphism group $\mathrm{Aut}(B)$. We shall overcome this difficulty by shifting our focus from the isomorphism relation on $R(\mathbb{Q}^n)$ to the coarser quasi-isomorphism relation.

Definition 3.5.1. *If $A, B \in R(\mathbb{Q}^n)$, then A and B are said to be* quasi-equal*, written $A \approx_n B$, if $A \cap B$ has finite index in both A and B.*

Definition 3.5.2. *If $A, B \in R(\mathbb{Q}^n)$, then A and B are said to be* quasi-isomorphic *if there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) \approx_n B$.*

The following result will play a key role in the proof of Theorem 3.3.2.

Theorem 3.5.3 (Thomas [34])**.** *The quasi-equality relation $\approx_n$ is a hyperfinite countable Borel equivalence relation.*

For each $A \in R(\mathbb{Q}^n)$, let $[A]$ be the $\approx_n$-class containing A. We shall consider the induced action of $GL_n(\mathbb{Q})$ on the set $X = \{[A] \mid A \in R(\mathbb{Q}^n)\}$ of $\approx_n$-classes. Of course, since $\approx_n$ is not smooth, X is not a standard Borel space; but fortunately this will not pose a problem in what follows. In order to describe the setwise stabilizer in $GL_n(\mathbb{Q})$ of each $\approx_n$-class $[A]$, we now make some further definitions.

Definition 3.5.4. *For each $A \in R(\mathbb{Q}^n)$, the ring of* quasi-endomorphisms *is*

$$QE(A) = \{\varphi \in \mathrm{Mat}_n(\mathbb{Q}) \mid (\exists m \geq 1)\, m\varphi \in \mathrm{End}(A)\}.$$

Clearly $\mathrm{QE}(A)$ is a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_n(\mathbb{Q})$, and so there are only countably many possibilities for $\mathrm{QE}(A)$, a fact which will be of crucial importance below.

Definition 3.5.5. $\mathrm{QAut}(A)$ *is the group of units of the* $\mathbb{Q}$*-algebra* $\mathrm{QE}(A)$*.*

Lemma 3.5.6 (Thomas [34])**.** *If* $A \in R(\mathbb{Q}^n)$*, then* $\mathrm{QAut}(A)$ *is the setwise stabilizer of* $[A]$ *in* $GL_n(\mathbb{Q})$*.*

For each $x \in (2)^G$, let $A_x = f(x) \in R(\mathbb{Q}^n)$. Since there are only countably many possibilities for the group $\mathrm{QAut}(A_x)$, there exists a *fixed* subgroup $L \leq GL_n(\mathbb{Q})$ and a Borel subset $X \subseteq (2)^G$ with $\mu(X) > 0$ such that $\mathrm{QAut}(A_x) = L$ for all $x \in X$. Since G acts ergodically on $((2)^G, \mu)$, it follows that $\mu(G \cdot X) = 1$. In order to simplify notation, we shall assume that $G \cdot X = (2)^G$. After slightly adjusting f if necessary, we can suppose that $\mathrm{QAut}(A_x) = L$ for all $x \in (2)^G$.

Notice that the quotient group $H = N_{GL_n(\mathbb{Q})}(L)/L$ acts freely on the corresponding set $Y = \{ [A] \mid \mathrm{QAut}(A) = L\}$ of $\approx_n$-classes. Furthermore, if $x \in (2)^G$ and $g \in G$, then there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A_x) = A_{g\cdot x}$ and it follows that $\varphi([A_x]) = [A_{g\cdot x}]$. Hence we can define a corresponding cocycle

$$\alpha : G \times (2)^G \to H$$

by setting

$$\alpha(g, x) = \text{ the unique } h \in H \text{ such that } h \cdot [A_x] = [A_{g\cdot x}].$$

Now let S be a countable simple nonamenable group which does not embed into any of the countably many possibilities for H. Applying Theorem 3.2.1, after deleting a null set and slightly adjusting f if necessary, we can suppose that

$$\alpha : G = SL_3(\mathbb{Z}) \times S \to H$$

is a group homomorphism. Since $S \leq \ker \alpha$, it follows that $f : (2)^G \to R(\mathbb{Q}^n)$ is a Borel homomorphism from the S-action on $(2)^G$ to the hyperfinite quasi-equality $\approx_n$-relation. Since S is nonamenable, the S-action on $(2)^G$ is E_0-ergodic and hence μ-almost all $x \in (2)^G$ are mapped to a single $\approx_n$-class, which is a contradiction. This completes the proof of Theorem 3.3.2.

4. Fourth Session

4.1. **Containment vs. Borel Reducibility.** Our next goal will be to present some applications of Ioana's Cocycle Superrigidity Theorem. We shall focus on a problem that was initially raised in the context of the Kechris Conjecture that the Turing equivalence relation $\equiv_T$ is universal. Recall that the translation action of the free group $\mathbb{F}_2$ on its power set gives rise to a universal countable Borel equivalence relation, which is denoted by E_∞. If we identify $\mathbb{F}_2$ with a suitably chosen group of recursive permutations of $\mathbb{N}$, then we see that E_∞ may be realized as a subset of $\equiv_T$. Thus the following conjecture of Hjorth [3] implies that $\equiv_T$ is universal.

Conjecture 4.1.1 (Hjorth)**.** *If F is a universal countable Borel equivalence relation on the standard Borel space X and E is a countable Borel equivalence relation such that $F \subseteq E$, then E is also universal.*

In [34], Thomas pointed out that it was not even known whether there existed a pair $F \subseteq E$ of countable Borel equivalence relations for which $F \not\leq_B E$. Soon afterwards, Adams [1] constructed a pair of countable Borel equivalence relations $F \subseteq E$ which were incomparable with respect to Borel reducibility. Most of this session will be devoted to a sketch of the proof of the following application of Ioana's Cocycle Superrigidity Theorem:

Theorem 4.1.2 (Thomas [33] 2002)**.** *There exists a pair of countable Borel equivalence relations $F \subseteq E$ on a standard Borel space X such that $E <_B F$.*

Here E and F will arise from the actions of $SL_n(\mathbb{Z})$ and a suitably chosen congruence subgroup on $SL_n(\mathbb{Z}_p)$. We shall first need to recall some basic facts about the ring $\mathbb{Z}_p$ of p-adic integers.

Definition 4.1.3. *The ring $\mathbb{Z}_p$ of p-adic integers is the inverse limit of the system*

$$\cdots \xrightarrow{\varphi_{n+1}} \mathbb{Z}/p^{n+1}\mathbb{Z} \xrightarrow{\varphi_n} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_1} \mathbb{Z}/p\mathbb{Z},$$

where $x + p^{n+1}\mathbb{Z} \overset{\varphi_n}{\longmapsto} x + p^n\mathbb{Z}$.

It is useful to think of the p-adic integers as formal sums

$$z = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots$$

where each $0 \leq a_n < p$. We define the *p-adic norm* $|\ \ |_p$ by

$$|z|_p = p^{-\operatorname{ord}_p(z)}, \qquad \operatorname{ord}_p(z) = \min\{n \mid a_n \neq 0\},$$

and the *p-adic metric* by

$$d_p(x, y) = |x - y|_p.$$

With this metric, $\mathbb{Z}_p$ is a compact Polish space having the integers $\mathbb{Z}$ as a dense subring. It follows that $SL_n(\mathbb{Z}_p)$ is a compact Polish group with dense subgroup $SL_n(\mathbb{Z}) \leq SL_n(\mathbb{Z}_p)$. Note that $SL_n(\mathbb{Z}_p)$ is the inverse limit of the system

$$\cdots \xrightarrow{\theta_{n+1}} SL_n(\mathbb{Z}/p^{n+1}\mathbb{Z}) \xrightarrow{\theta_n} SL_n(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\theta_{n-1}} \cdots \xrightarrow{\theta_1} SL_n(\mathbb{Z}/p\mathbb{Z}),$$

where θ_n is the map induced by φ_n.

Since $SL_n(\mathbb{Z}_p)$ is compact, there exists a unique Haar probability measure on $SL_n(\mathbb{Z}_p)$; i.e. a unique probability measure μ_p which is invariant under the left translation action of $SL_n(\mathbb{Z}_p)$ on itself.[2] In fact, μ_p is simply the inverse limit of the counting measures on

$$\cdots \xrightarrow{\theta_{n+1}} SL_n(\mathbb{Z}/p^{n+1}\mathbb{Z}) \xrightarrow{\theta_n} SL_n(\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\theta_{n-1}} \cdots \xrightarrow{\theta_1} SL_n(\mathbb{Z}/p\mathbb{Z}).$$

Observe that if $H \leq SL_n(\mathbb{Z}_p)$ is an open subgroup, then H has finite index in $SL_n(\mathbb{Z}_p)$ and

$$\mu_p(H) = \frac{1}{[SL_n(\mathbb{Z}_p) : H]}.$$

Theorem 4.1.4. *μ_p is the unique $SL_n(\mathbb{Z})$-invariant probability measure on $SL_n(\mathbb{Z}_p)$.*

Proof. First note that $SL_n(\mathbb{Z}_p)$ acts continuously on the space $\mathcal{M}$ of probability measures on $SL_n(\mathbb{Z}_p)$. It follows that if ν is any probability measure on $SL_n(\mathbb{Z}_p)$, then

$$S_\nu = \{g \in SL_n(\mathbb{Z}_p) \mid \nu \text{ is } g\text{-invariant}\}$$

is a closed subgroup of $SL_n(\mathbb{Z}_p)$. Hence, since $SL_n(\mathbb{Z})$ is a dense subgroup of $SL_n(\mathbb{Z}_p)$, any $SL_n(\mathbb{Z})$-invariant probability measure is actually $SL_n(\mathbb{Z}_p)$-invariant and thus must be μ_p. $\square$

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[2]The Haar measure on a compact group is also invariant under the right translation action.

4.2. **Unique Ergodicity and Ergodic Components.** An action of a group G on a standard Borel G-space X is said to be *uniquely ergodic* iff there exists a unique G-invaraint probability measure μ on X. In this case, it is well-known that μ must be ergodic. To see this, suppose that $A \subseteq X$ is a G-invariant Borel set with $0 < \mu(A) < 1$. Then we can define distinct G-invariant probability measures by

$$\begin{aligned} \nu_1(Z) &= \mu(Z \cap A)/\mu(A) \\ \nu_2(Z) &= \mu(Z \setminus A)/\mu(X \setminus A), \end{aligned}$$

which is a contradiction. Note that Theorem 4.1.4 simply states that the action of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$ is uniquely ergodic.

Next suppose that Γ is a countable group and that $\Lambda \leq \Gamma$ is a subgroup of finite index. Let X be a standard Borel Γ-space with an invariant ergodic probability measure μ. Then a Λ-invariant Borel set $Z \subseteq X$ with $\mu(Z) > 0$ is said to be an *ergodic component* for the action of Λ on X iff Λ acts ergodically on (Z, μ_Z), where μ_Z is the normalized probability measure on Z defined by $\mu_Z(A) = \mu(A)/\mu(Z)$. It is easily checked that there exists a partition $Z_1 \sqcup \cdots \sqcup Z_d$ of X into finitely many ergodic components and that the collection of ergodic components is uniquely determined up to μ-null sets. Furthermore, if the action of Γ on X is uniquely ergodic, then the action of Λ on each ergodic component is also uniquely ergodic.

Now let $n \geq 3$ and fix some prime p. Consider the left translation action of the subgroup $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$. Then we have already seen that this action is uniquely ergodic. Let $\Lambda = \ker \varphi$ and $H = \ker \psi$ be the kernels of the homomorphisms

$$\varphi : SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/p\mathbb{Z})$$

and

$$\psi : SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_p/p\mathbb{Z}_p) \cong SL_n(\mathbb{Z}/p\mathbb{Z}).$$

Then H is the closure of Λ in $SL_n(\mathbb{Z}_p)$ and the ergodic decomposition of the Λ-action coincides with the coset decomposition

$$SL_n(\mathbb{Z}_p) = Hg_1 \sqcup \cdots \sqcup Hg_d, \qquad d = |SL_n(\mathbb{Z}/p\mathbb{Z})|.$$

We are now ready to state Thomas' result.

Theorem 4.2.1. *Let $n \geq 3$ and let $F \subseteq E$ be the orbit equivalence relations of the actions of Λ and $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$. Then $E <_B F$.*

We shall devote the next section to a proof of this result.

4.3. **The proof of Theorem 4.2.1.** By considering the ergodic decomposition of the Λ-action,

$$SL_n(\mathbb{Z}_p) = Hg_1 \sqcup \cdots \sqcup Hg_d, \qquad d = |SL_n(\mathbb{Z}/p\mathbb{Z})|,$$

we see that

$$F = E_1 \oplus \cdots \oplus E_d, \qquad \text{where } E_i = F \restriction Hg_i.$$

We claim that $E \sim_B E_i$ for each $1 \leq i \leq d$.

To see that $E_i \leq_B E$, we check that the inclusion map $Hg_i \to SL_n(\mathbb{Z}_p)$ is a Borel reduction. Suppose that $x, y \in Hg_i$. Clearly if xE_iy then xEy, since $F \subseteq E$. Conversely, if xEy, then there exists $\gamma \in SL_n(\mathbb{Z})$ such that $\gamma x = y$, whence $\emptyset \neq \gamma Hg_i \cap Hg_i = H\gamma g_i \cap Hg_i$ and so $\gamma \in SL_n(\mathbb{Z}) \cap H = \Lambda$.

To show that $E \leq_B E_i$, we choose the coset representatives g_k so that each $g_k \in SL_n(\mathbb{Z})$. Then for each $1 \leq k \leq d$, define $h_k : Hg_k \to Hg_i$ by $h_k(x) = g_i g_k^{-1} x$. We claim that $h = h_1 \cup \cdots \cup h_d$ is a Borel reduction from E to E_i. To see this, note that if $x, y \in SL_n(\mathbb{Z}_p)$, then

$$\begin{aligned} xEy \quad &\text{iff} \quad h(x)Eh(y) \\ &\text{iff} \quad h(x)E_ih(y), \end{aligned}$$

where this last equivalence holds because $h(x), h(y) \in Hg_i$. This completes the proof that $E \sim_B E_i$ for each $1 \leq i \leq d$; and hence we have that

$$F \sim_B \underbrace{E \oplus \cdots \oplus E}_{d \text{ times}}.$$

Therefore it will be enough to prove the following:

Theorem 4.3.1 (Thomas [33] 2002)**.** *If $n \geq 3$, then*

$$E <_B E \oplus E <_B \cdots <_B \underbrace{E \oplus \cdots \oplus E}_{m \text{ times}} <_B \cdots$$

Proof. Let $\Gamma = SL_n(\mathbb{Z})$ and let $(K, \mu) = (SL_n(\mathbb{Z}_p), \mu_p)$, so that E is the orbit equivalence relation arising from the action of Γ on K. It clearly suffices to show that if $f : K \to K$ is a Borel reduction from E to E, then $\mu(\Gamma \cdot f(K)) = 1$.

So suppose that $f : K \to K$ is a Borel reduction from E to E. Since Γ acts freely on K, we can define a corresponding Borel cocycle $\alpha : \Gamma \times K \to \Gamma$ by

$$\alpha(g, x) = \text{ the unique } h \in \Gamma \text{ such that } h \cdot f(x) = f(g \cdot x).$$

By Ioana's Superrigidity Theorem [18] (which we will state and discuss in the next section), there exists a subgroup $\Delta \leq \Gamma$ of finite index and an ergodic component $X \subseteq K$ for the Δ-action such that $\alpha \restriction (\Delta \times X)$ is equivalent to a group homomorphism

$$\psi : \Delta \to SL_n(\mathbb{Z}).$$

After slightly adjusting f if necessary, we can suppose that $\alpha \restriction (\Delta \times X) = \psi$ and hence that

$$\psi(g) \cdot f(x) = f(g \cdot x) \quad \text{for all } g \in \Delta \text{ and } x \in X.$$

Furthermore, since Δ is residually finite, after passing to a subgroup of finite index if necessary, we can also suppose that $\Delta \cap Z(SL_n(\mathbb{Z})) = 1$.

Claim 4.3.2. *Either $\psi(\Delta)$ is finite, or else ψ is an embedding and $\psi(\Delta)$ is a subgroup of finite index in $SL_n(\mathbb{Z})$.*

Proof of Claim 4.3.2. Suppose that ψ is not an embedding and let $N = \ker \psi$. Then the Margulis Normal Subgroup Theorem [26, Chapter VIII] implies that $[\,\Delta : N\,] < \infty$ and hence $\psi(\Delta)$ is finite. Thus we can suppose that ψ is an embedding. Let $\pi : SL_n(\mathbb{Z}) \to PSL_n(\mathbb{Z})$ be the canonical surjective homomorphism and let $\theta = \pi \circ \psi$. Then, arguing as above, we see that $\theta : \Delta \to PSL_n(\mathbb{Z})$ is also an embedding. Applying Margulis [26, Chapter VII], it follows that θ extends to an $\mathbb{R}$-rational homomorphism $\Theta : SL_n(\mathbb{R}) \to PSL_n(\mathbb{R})$ and it is easily seen that Θ is surjective. Since Δ is a lattice in $SL_n(\mathbb{R})$, it follows that $\theta(\Delta) = \Theta(\Delta)$ is a lattice in $PSL_n(\mathbb{R})$ and this implies that $\theta(\Delta)$ is a subgroup of finite index in $PSL_n(\mathbb{Z})$. Hence $\psi(\Delta)$ is a subgroup of finite index in $SL_n(\mathbb{Z})$. □

First suppose that $\psi(\Delta)$ is finite. Then we can define a Δ-invariant map $\phi : X \to [K]^{<\omega}$ by

$$\phi(x) = \{ f(g \cdot x) \mid g \in \Delta \};$$

and since Δ acts ergodically on X, it follows that ϕ is constant on a μ-conull subset of X, which is a contradiction.

Thus ψ is an embedding and $\psi(\Delta)$ is a subgroup of finite index in $SL_n(\mathbb{Z})$. Let $Y_1, \dots, Y_d$ be the ergodic components for the action of $\psi(\Delta)$ on K. Since Δ acts ergodically on X, we can suppose that there exists a fixed $Y = Y_i$ such that $f : X \to Y$. Recalling that $\psi(g) \cdot f(x) = f(g \cdot x)$, we can now define a $\psi(\Delta)$-invariant probability measure ν on Y by

$$\nu(Z) = \mu(f^{-1}(Z))/\mu(X).$$

Since the action of $\psi(\Delta)$ on Y is uniquely ergodic, it follows that $\nu(Z) = \mu(Z)/\mu(Y)$. Hence $\mu(f(X)) = \mu(Y) > 0$ and so $\mu(\Gamma \cdot f(K)) = 1$, as desired. This completes the proof of Theorem 4.3.1, and hence also that of Theorem 4.2.1. □

4.4. **Profinite Actions and Ioana Superrigidity.**

Definition 4.4.1. *Suppose that Γ is a countable group and that X is a standard Borel Γ-space with invariant probability measure μ. Then the action of Γ on (X, μ) is said to be* profinite *if there exists a directed system of finite Γ-spaces X_n with invariant probability measures μ_n such that*

$$(X, \mu) = \varprojlim (X_n, \mu_n).$$

For example, suppose that K is a profinite group and that $\Gamma \leq K$ is a countable dense subgroup. If $L \leq K$ is a closed subgroup, then the action of Γ on K/L is profinite. In particular, if Γ is a residually finite group and

$$\Gamma = \Gamma_0 > \Gamma_1 > \cdots > \Gamma_n > \cdots$$

is a decreasing sequence of finite index normal subgroups such that $\bigcap \Gamma_n = 1$, then Γ is a dense subgoup of the profinite group $\varprojlim \Gamma/\Gamma_n$ and its action as a subgroup will be profinite. Of course, this example covers the situation discussed above; i.e. the action of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$ is profinite.

We are now ready to state Ioana's Cocycle Superrigidity Theorem [18], which was used in our proof of Theorem 4.2.1.

Theorem 4.4.2 (Ioana)**.** *Let Γ be a countably infinite Kazhdan group and let (X, μ) be a free ergodic profinite Γ-space. Suppose that H is any countable group and that $\alpha : \Gamma \times X \to H$ is a Borel cocycle. Then there exists a subgroup $\Delta \leq \Gamma$ of finite index and an ergodic component $Y \subseteq X$ for the Δ-action such that $\alpha \restriction (\Delta \times Y)$ is equivalent to a homomorphism $\psi : \Delta \to H$.*

To conclude this section, we shall present a final application of Ioana's theorem.

Theorem 4.4.3 (Thomas [33] 2002)**.** *Fix $n \geq 3$. For each nonempty set S of primes, regard $SL_n(\mathbb{Z})$ as a subgroup of*

$$G(S) = \prod_{p \in S} SL_n(\mathbb{Z}_p)$$

via the diagonal embedding and let E_S be the corresponding orbit equivalence relation. If $S \neq T$, then E_S and E_T are incomparable with respect to Borel reducibility.

Sketch Proof. For simplicity, suppose that $S = \{p\}$ and $T = \{q\}$, where $p \neq q$ are distinct primes. Suppose that $f : SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_q)$ is a Borel reduction from $E_{\{p\}}$ to $E_{\{q\}}$. Then applying Ioana Superrigidity and arguing as in the proof of Theorem 4.3.1, we see that after passing to subgroups of finite index and ergodic components if necessary,

$$(SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p) \cong (SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_q), \mu_q)$$

as measure-preserving permutation groups. Hence it only remains to detect the prime p in $(SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p)$.

Towards this end, recall that $\mathrm{Aut}(SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p)$ consists of the measure-preserving bijections $\varphi : SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{Z}_p)$ such that for all $\gamma \in SL_n(\mathbb{Z})$,

$$\varphi(\gamma \cdot x) = \gamma \cdot \varphi(x) \quad \text{for } \mu_p\text{-a.e. } x,$$

where we identify two such maps if they agree μ_p-a.e. Notice that for each $g \in SL_n(\mathbb{Z}_p)$, we can define a corresponding automorphism $\varphi \in \mathrm{Aut}(SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p)$ by $\varphi(x) = xg$. (Here we have made use of the fact that the Haar measure μ_p on the compact group $SL_n(\mathbb{Z}_p)$ is also invariant under the right translation action.) The following proposition shows that there are no others.

Proposition 4.4.4 (Gefter-Golodets [13]). *$Aut(SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p) = SL_n(\mathbb{Z}_p)$.*

Proof. Let $\varphi \in \mathrm{Aut}(SL_n(\mathbb{Z}), SL_n(\mathbb{Z}_p), \mu_p)$. For each $x \in SL_n(\mathbb{Z}_p)$, let $h(x) \in SL_n(\mathbb{Z}_p)$ be such that $\varphi(x) = xh(x)$. If $\gamma \in SL_n(\mathbb{Z})$, then for μ_p-a.e. x,

$$\varphi(\gamma \cdot x) = \gamma \cdot \varphi(x) = \gamma \cdot xh(x)$$

and so $h(\gamma \cdot x) = h(x)$. Since $SL_n(\mathbb{Z})$ acts ergodically on $(SL_n(\mathbb{Z}_p), \mu_p)$, there exists a fixed $g \in SL_n(\mathbb{Z}_p)$ such that $h(x) = g$ for μ_p-a.e. x. □

Thus we have reduced our problem to that of detecting the prime p in the topological group $SL_n(\mathbb{Z}_p)$. But this is easy, since $SL_n(\mathbb{Z}_p)$ is virtually a pro-p group. More precisely, if H is any open subgroup of $SL_n(\mathbb{Z}_p)$, then

$$[SL_n(\mathbb{Z}_p) : H] = bp^l$$

for some $l \geq 0$ and some divisor b of $|SL_n(\mathbb{Z}/p\mathbb{Z})|$. This completes our sketch of a proof of Theorem 4.4.3. □

4.5. **Open Problems.** In this closing section, we shall point out some of the many open problems in the field of countable Borel equivalence relations.

4.5.1. *Hyperfinite Relations.* Recall that a countable Borel equivalence relation E on a standard Borel space X is said to be *hyperfinite* if E can be written as the union of a countable increasing sequence of *finite* Borel equivalence relations. A theorem of Dougherty-Jackson-Kechris [7] provides two additional characterizations:

Theorem 4.5.1 (Dougherty-Jackson-Kechris)**.** *If E is a countable Borel equivalence relation on a standard Borel space X, then the following are equivalent:*

- *E is hyperfinite.*
- *$E \leq_B E_0$.*
- *There exists a Borel action of $\mathbb{Z}$ on X such that $E = E_{\mathbb{Z}}^X$.*

In fact, *every* $\mathbb{Z}$-action on a standard Borel $\mathbb{Z}$-space X yields a hyperfinite orbit equivalence relation; and by a recent theorem of Gao-Jackson [12], even more is true.

Theorem 4.5.2 (Gao-Jackson)**.** *If G is a countable abelian group and X is a standard Borel G-space, then E_G^X is hyperfinite.*

An important question concerns how much further this result can be extended. By a theorem of Jackson-Kechris-Louveau [19], if G is a countable, nonamenable group, then the orbit equivalence relation E_G arising from the free action of G on $((2)^G, \mu)$ is *not* hyperfinite. However, the following problem remains open:

Question 4.5.3 (Weiss [38])**.** *Suppose that G is a countable amenable group and that X is a standard Borel G-space. Does it follow that E_G^X is hyperfinite?*

As a partial answer, we have the following theorem of Connes-Feldman-Weiss [5].

Theorem 4.5.4 (Connes-Feldman-Weiss)**.** *Suppose that G is a countable amenable group and that X is a standard Borel G-space. If μ is* any *Borel probability measure on X, then there exists a Borel subset $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is hyperfinite.*

4.5.2. *Treeable Relations.*

Definition 4.5.5. *The countable Borel equivalence relation E on X is said to be* treeble *iff there is an acyclic Borel graph (X, R) whose connected components are the E-classes.*

For example, if a countable free group $\mathbb{F}$ acts freely on a standard Borel $\mathbb{F}$-space X, then the corresponding orbit equivalence relation $E_{\mathbb{F}}^{X}$ is treeable. Conversely, by a theorem of Jackson-Kechris-Louveau [19], if E is treeable, then there exists a free Borel action of a countable free group $\mathbb{F}$ on a standard Borel space Y such that $E \sim_B E_{\mathbb{F}}^{Y}$. It is easily seen that every hyperfinite countable Borel equivalence relation is treeable; and it is known that the universal countable Borel equivalence relation E_∞ is *not* treeable. On the other hand, there exist countable Borel equivalence relations which are treeable but not hyperfinite. For example, the orbit equivalence relation $E_{\infty T}$ arising from the free action of $\mathbb{F}_2$ on $(2)^{\mathbb{F}_2}$ is not hyperfinite.

Theorem 4.5.6 (Jackson-Kechris-Louveau [19]). *$E_{\infty T}$ is universal for treeable countable Borel equivalence relations.*

For many years, it was an important open problem whether there existed infinitely many treeable countable Borel equivalence relations up to Borel bireducibility. This question has very recently been solved by Hjorth:

Theorem 4.5.7 (Hjorth [16]). *There exist uncountably many treeable countable Borel equivalence relations which are pairwise incomparable with respect to Borel reducibility.*

An intriguing aspect of Hjorth's proof is that it does not provide an *explicit example* of a single pair E, F of incomparable treeable countable Borel equivalence relations. However, there is a natural candidate for an explicit family of uncountably many treeable countable Borel equivalence relations which are pairwise incomparable with respect to Borel reducibility. For each nonempty set S of primes, regard $SL_2(\mathbb{Z})$ as a subgroup of

$$G(S) = \prod_{p \in S} SL_2(\mathbb{Z}_p)$$

via the diagonal embedding and let E_S be the corresponding orbit equivalence relation. Then E_S is a non-hyperfinite profinite treeable Borel equivalence relation.

Conjecture 4.5.8 (Thomas). *If $S \neq T$, then E_S and E_T are incomparable with respect to Borel reducibility.*

We will finish this section with an attractive (but almost certainly false) conjecture of Kechris. Recall that in 1980, Ol'shanskii [28] refuted the so-called "von Neumann conjecture" (which is actually due to Day) by constructing a periodic nonamenable group, which clearly had no free nonabelian subgroups. However, the following analogous problem remains open:

Conjecture 4.5.9 (Kechris)**.** *If E is a non-hyperfinite countable Borel equivalence relation, then there exists a non-hyperfinite treeable countable Borel equivalence relation F such that $F \leq_B E$.*

4.5.3. *Universal Relations.* There are many basic open problems concerning universal countable Borel equivalence relations, including the following:

Conjecture 4.5.10 (Hjorth)**.** *If E is a universal countable Borel equivalence relation on the standard Borel space X and F is a countable Borel equivalence relation such that $E \subseteq F$, then F is also universal.*

Conjecture 4.5.11 (Kechris)**.** *The Turing equivalence relation $\equiv_T$ is countable universal.*

Question 4.5.12 (Jackson-Kechris-Louveau [19])**.** *Suppose that E is a universal countable Borel equivalence relation on the standard Borel space X and that $Y \subseteq X$ is an E-invariant Borel subset. Does it follow that either $E \restriction Y$ or $E \restriction (X \setminus Y)$ is universal?*

Finally we conclude with two questions concerning the notion of a *minimal cover* of an equivalence relation.

Definition 4.5.13. *If E, E' are countable Borel equivalence relations, then E' is a minimal cover of E if:*

- *$E <_B E'$; and*
- *if F is a countable Borel equivalence relation such that $E \leq_B F \leq_B E'$, then either $E \sim_B F$ or $F \sim_B E'$.*

Open Problem 4.5.14 (Thomas)**.** *Find an example of a nonsmooth countable Borel equivalence relation which has a minimal cover.*

Open Problem 4.5.15 (Thomas)**.** *Find an example of a nonuniversal countable Borel equivalence relation which does* not *have a minimal cover.*

References

[1] S. Adams, *Containment does not imply Borel reducibility*, in: Set Theory: The Hajnal Conference (Ed: S. Thomas), DIMACS Series, vol. 58, American Mathematical Society, 2002, pp. 1-23.

[2] S. Adams and A. S. Keckris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), 909-943.

[3] A. Andretta, R. Camerlo, and G. Hjorth, *Conjugacy equivalence relations on subgroups*, Fund. Math., **167** (2001), 189-212.

[4] R. Baer, *Abelian groups without elements of finite order*, Duke Math. Journal, **3** (1937), 68-122.

[5] A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory and Dynamical Systems, **1** (1981), 430-450.

[6] C. Champetier, *L'espace des groupes de type fini*, Topology **39** (2000), 657–680.

[7] R. Dougherty, S. Jackson, and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341 (1)** (1994), 193-225.

[8] R. Dougherty and A. S. Kechris, *How many Turing degrees are there?*, in: Computability Theory and its Applications (Boulder, CO, 1999), Contemp. Math. **257**, Amer. Math. Soc., 2000, 83–94.

[9] M. J. Dunwoody and A. Pietrowski, *Presentations of the trefoil group*, Canad. Math. Bull. **16** (1973), 517–520.

[10] J. Feldman and C. C. Moore, *Ergodic equivalence relations and Von Neumann algebras I*, Trans. Amer. Math. Soc. **234** (1977), 289-324.

[11] H. Friedman and L. Stanley, *A Borel reducibility theory for classes of countable structures* J. Symbolic Logic **54** (1989), 894–914.

[12] S. Gao and S. Jackson, *Countable abelian group actions and hyperfinite equivalence relations*, preprint, 2007.

[13] S. L. Gefter and V. Ya. Golodets, *Fundamental groups for ergodic actions and actions with unit fundamental groups*, Publ. RIMS, Kyoto Univ. **24** (1988), 821-847.

[14] L. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dechotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3 (4)** (1990), 903-928.

[15] G. Hjorth, *Around nonclassifiability for countable torsion-free abelian groups*, in Abelian groups and modules (Dublin, 1998), Trends Math., Birkhäuser, Basel (1999), 269-292.

[16] G. Hjorth, *Treeable equivalence relations*, preprint, 2008.

[17] G. Hjorth and A. S. Kechris, *Borel equivalence relations and classification of countable models*, Annals of Pure and Applied Logic, **82** (1996), 221-272.

[18] A. Ioana, *Cocycle superrigidity for profinite actions of Kazhdan groups*, in progress.

[19] S. Jackson, A. S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, J. Math. Logic, **2** (2002), 1-80.

[20] V. F. R. Jones and K. Schmidt, *Asymptotically invariant sequences and approximate finiteness*, Amer. J. Math. **109** (1987), 91-114.

[21] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics **156** Springer-Verlag, New York, 1994.

[22] K. Kuratowski, *Sur une généralisation de la notion d'homéomorphie*, Fund. Math. **22** (1934), 206-220.

[23] A. G. Kurosh, *Primitive torsionsfreie abelsche Gruppen vom endlichen Range*, Ann. Math. 38 (1937), 175-203.

[24] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, Progress in Mathematics **125**, Birkhäuser, 1994.

[25] A. I. Malcev, *Torsion-free abelian groups of finite rank* (Russian), Mat. Sbor. **4** (1938), 45-68.

[26] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Erg. der Math. und ihrer Grenz. **17**, Springer-Verlag, 1991.

[27] B. D. Miller, *Borel equivalence relations and everywhere faithful actions of free products*, preprint, 2006.

[28] A. Yu. Ol'shanskii, *On the question of the existence of an invariant mean on a group* (Russian), Uspekhi Mat. Nauk, **35** (1980), no. 4, 199-200.

[29] S. Popa, *Cocycle and orbit equivalence superrigidity for malleable actions of ω-rigid groups*, Inventiones Mathematicae, **170 (2)** (2007), 243-295.

[30] J. Silver, *Counting the number of equivalence classes of Borel and co-analytic equivalence relations*, Ann. Math. Logic, **18** (1980), 1-28.

[31] T. Slaman and J. Steel, *Definable functions on degrees*, in: Cabal Seminar 81-85, Lecture Notes in Mathematics, **1333**, Springer-Verlag (1998), 37-55

[32] S. M. Srivastava, *A Course on Borel Sets*, Graduate Texts in Mathematics **180**, Springer-Verlag, 1998.

[33] S. Thomas, *Superrigidity and countable Borel equivalence relations*, Annals Pure Appl. Logic, **120** (2003), 237-262.

[34] S. Thomas, *The classification problem for torsion-free abelian groups of finite rank*, J. Amer. Math. Soc. **16** (2003), 233-258.

[35] S. Thomas, *On the complexity of the quasi-isometry and virtual isomorphism problems for finitely generated groups*, Groups Geom. Dyn. **2** (2008), 281–307.

[36] S. Thomas, *Popa superrigidity and countable Borel equivalence relations*, Ann. Pure Appl. Logic **158** (2009), 175-189.

[37] S. Wagon, *The Banach-Tarski Paradox*, Encyclopedia of Mathematics and its Applications **24**, Cambridge University Press, 1985.

[38] B. Weiss, *Measureable dynamics*, Conference in modern analysis and probability (R. Beals et al., editors), Contemporary Mathematics, vol. 26, American Mathematical Society, Providence, Rhode Island, 1984, 395-421.

Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019, USA