PROPER FORCING REMASTERED

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Abstract. We present the method introduced by Neeman of generalized side conditions with two types of models. We then discuss some applications: a variation of the Friedman-Mitchell poset for adding a club with finite conditions, the consistency of the existence of an ω_2 increasing chain in $(\omega_1^{\omega_1}, <_{\rm fin})$, originally proved by Koszmider, and the existence of a thin very tall superatomic Boolean algebra, originally proved by Baumgartner-Shelah. We expect that the present method will have many more applications.

Introduction

We present a generalization of the method of model as side conditions. Generally speaking a poset that uses models as side conditions is a notion of forcing whose elements are pairs, consisting of a working part which is some partial information about the object we wish to add and a finite $\in$-chain of countable elementary substructures of $H(\theta)$, for some cardinal θ i.e. the structure consisting of sets whose transitive closure has cardinality less than θ. The models in the side condition are used to control the extension of the working part. This is crucial in showing some general property of the forcing such as properness.

The generalization we now present amounts to allowing also certain uncountable models in the side conditions. This is used to show that the forcing preserves both $\aleph_1$ and $\aleph_2$. This approach was introduced by Neeman [9] who used it to give an alternative proof of the consistency of PFA and also to obtain generalizations of PFA to higher cardinals. In §1 we present the two-type poset of pure side conditions from [9], in the case of countable models and approachable models of size ω_1, and work out the details of some of its main properties that were mentioned in [9]. The remainder of the paper is devoted to applications. We will be primarily interested in adding certain combinatorial objects of size $\aleph_2$. These results were known by other methods but we believe that the present method is more efficient and will have other applications. In §2 we present a version of the forcing for adding a club in ω_2 with finite conditions, preserving

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ω_1 and ω_2. This fact has been shown to be consistent with ZFC independently by Friedman ([3]) and Mitchell ([7]) using more complicated notions of forcing. In §3 we show how to add a chain of length ω_2 in the structure $(\omega_1^{\omega_1}, <_{\mathrm{fin}})$. This result is originally due to Koszmider [5]. Finally, in §4 we give another proof of a result of Baumgartner and Shelah [2] by using side condition forcing to add a thin very tall superatomic Boolean algebra.

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1. The forcing $\mathbb{M}$

In this section we present the forcing consisting of pure side conditions. Our presentation follows [9], but we only consider side conditions consisting of models which are either countable or of size $\aleph_1$. We consider the structure $(H(\aleph_2), \in, \trianglelefteq)$ equipped with a fixed well-ordering $\trianglelefteq$. In this way we have definable Skolem functions, so if M and N are elementary submodels of $H(\aleph_2)$ then so is $M \cap N$.

Definition 1.1. *Let P an elementary submodel of $H(\aleph_2)$ of size $\aleph_1$. We say that P is* internally approachable *if it can be written as the union of an increasing continuous $\in$-chain $\langle P_\xi : \xi < \omega_1 \rangle$ of countable elementary submodels of $H(\aleph_2)$ such that $\langle P_\xi : \xi < \eta \rangle \in P_{\eta+1}$, for every ordinal $\eta < \omega_1$.*

If P is internally approachable of size $\aleph_1$ we let $\vec{P}$ denote the least $\trianglelefteq$-chain witnessing this fact and we write P_ξ for the ξ-th element of this chain. Note also that in this case $\omega_1 \subseteq P$.

Definition 1.2. *We let $\mathcal{E}_0^2$ denote the collection of all countable elementary submodels of $H(\aleph_2)$ and $\mathcal{E}_1^2$ the collection of all internally approachable elementary submodels of $H(\aleph_2)$ of size $\aleph_1$. We let $\mathcal{E}^2 = \mathcal{E}_0^2 \cup \mathcal{E}_1^2$.*

The following fact is well known.

Fact 1.3. *The set $\mathcal{E}_1^2$ is stationary in $[H(\aleph_2)]^{\aleph_1}$.* □

We are now ready to define the forcing notion $\mathbb{M}$ consisting of pure side conditions.

Definition 1.4. *The forcing notion $\mathbb{M}$ consists of finite $\in$-chains $p = \mathcal{M}_p$ of models in $\mathcal{E}^2$ closed under intersection. The order on $\mathbb{M}$ is reverse inclusion, i.e. $q \leq p$ if $\mathcal{M}_p \subseteq \mathcal{M}_q$.*

Suppose M and N are elements of $\mathcal{E}^2$ with $M \in N$. If $|M| \leq |N|$ then $M \subseteq N$. However, if M is of size $\aleph_1$ and N is countable then the $\trianglelefteq$-least chain $\vec{M}$ witnessing that M is internally approachable belongs to N and so $M \cap N = M_{\delta_N}$, where $\delta_N = N \cap \omega_1$ and M_{δ_N} is the δ_N-th member of $\vec{M}$.

We can split every condition in $\mathbb{M}$ in two parts: the models of size $\aleph_0$ and the models of size $\aleph_1$.

Definition 1.5. *For $p \in \mathbb{M}$ let $\pi_0(p) = p \cap \mathcal{E}_0^2$ and $\pi_1(p) = p \cap \mathcal{E}_1^2$.*

Let us see some structural property of the elements of $\mathbb{M}$. First, let $\in^*$ be the transitive closure of the $\in$ relation, i.e. $x \in^* y$ if $x \in \mathrm{tcl}(y)$. Clearly, if $p \in \mathbb{M}$ then $\in^*$ is a total ordering on $\mathcal{M}_p$. Given $M, N \in \mathcal{M}_p \cup \{\emptyset, H(\aleph_2)\}$ with $M \in^* N$ let

$$(M, N)_p = \{P \in \mathcal{M}_p : M \in^* P \in^* N\}.$$

We let $(M, N]_p = (M, N)_p \cup \{N\}$, $[M, N)_p = (M, N)_p \cup \{M\}$ and $[M, N]_p = (M, N)_p \cup \{M, N\}$. Given a condition $p \in \mathbb{M}$ and $M \in p$ we let $p \restriction M$ denote the restriction of p to M, i.e. $\mathcal{M}_p \cap M$.

Fact 1.6. *Suppose $p \in \mathbb{M}$ and $N \in \pi_1(p)$. Then $\mathcal{M}_p \cap N = (\emptyset, N)_p$. Therefore, $p \cap N \in \mathbb{M}$.* □

Fact 1.7. *Suppose $p \in \mathbb{M}$ and $M \in \pi_0(p)$. Then*

$$\mathcal{M}_p \cap M = \mathcal{M}_p \setminus \bigcup \{[M \cap N, N)_p : N \in (\pi_1(p) \cap M) \cup \{H(\aleph_2)\}\}.$$

Therefore, $p \cap M \in \mathbb{M}$. □

The next lemma will be used in the proof of properness of $\mathbb{M}$.

Lemma 1.8. *Suppose $M \in \mathcal{E}^2$ and $p \in \mathbb{M} \cap M$. Then there is a new condition p^M, which is the smallest element of $\mathbb{M}$ extending p and containing M as an element.*

Proof. If $M \in \mathcal{E}_1^2$ we can simply let

$$p^M = \mathcal{M}_{p^M} = \mathcal{M}_p \cup \{M\}.$$

If $M \in \mathcal{E}_0^2$ we close $\mathcal{M}_p \cup \{M\}$ under intersections and show that it is still an $\in$-chain. First of all notice that, since p is finite and belongs to M, we have $\mathcal{M}_p \subseteq M$. For this reason if $P \in \pi_0(p)$, then $P \cap M = P$. On the other hand, if $P \in \pi_1(p)$, by the internal approachability of P and the fact that $P \in M$ we have that $P \cap M \in P$. Now, if $N \in P$ is the $\in^*$-greatest element of $\mathcal{M}_p$ below P, then $N \in P \cap M$, since $\mathcal{M}_p \subseteq M$. Finally the $\in^*$-greatest element of $\mathcal{M}_p$ belongs to M, since $\mathcal{M}_p$ does. □

Let $\mathcal{P}$ be a forcing notion. We say that a set M is *adequate* for $\mathcal{P}$ if for every $p, q \in M \cap \mathcal{P}$ if p and q are compatible then there is $r \in \mathcal{P} \cap M$ such that $r \leq p, q$. Note that we do not require that $\mathcal{P}$ belongs to M. In the forcing notions we consider if two conditions p and q are compatible then this will be witnessed by a condition r which is Σ_0-definable from p and q. Thus, all elements of $\mathcal{E}^2$ will be adequate for the appropriate forcing notions.

Definition 1.9. *Suppose $\mathcal{P}$ is a forcing notion and M is adequate for $\mathcal{P}$. We say that a condition p is $(M,\mathcal{P})$*-strongly generic *if p forces that $\dot{G}\cap M$ is a V-generic subset of $\mathcal{P}\cap M$, where $\dot{G}$ is the canonical name for the V-generic filter over $\mathcal{P}$.*

In order to check that a condition is strongly generic over a set M we can use the following characterization, see [8] for a proof.

Fact 1.10. *Suppose $\mathcal{P}$ a notion of forcing and M is adequate for $\mathcal{P}$. A condition p is $(M,\mathcal{P})$-strongly generic if and only if for every $r \leq p$ in $\mathcal{P}$ there is a condition $r|M \in \mathcal{P}\cap M$ such that any condition $q \leq r|M$ in M is compatible with r.*

Definition 1.11. *Suppose $\mathcal{P}$ is a forcing notion and $\mathcal{S}$ is a collection of sets adequate for $\mathcal{P}$. We say that $\mathcal{P}$ is $\mathcal{S}$*-strongly proper*, if for every $M \in \mathcal{S}$, every condition $p \in \mathcal{P}\cap M$ can be extended to an $(M,\mathcal{P})$-strongly generic condition q.*

Our goal is to show that $\mathbb{M}$ is $\mathcal{E}^2$-strongly proper. We will need the following.

Lemma 1.12. *Suppose $r \in \mathbb{M}$ and $M \in \mathcal{M}_r$. Let $q \in M$ be such that $q \leq r \cap M$. Then q and r are compatible.*

Proof. If M is uncountable then one can easily check that $\mathcal{M}_s = \mathcal{M}_q \cup \mathcal{M}_r$ is an $\in$-chain which is closed under intersection. Therefore $s = \mathcal{M}_s$ is a common extension of q and r. Suppose now M is countable. We first check that $\mathcal{M}_q \cup \mathcal{M}_r$ is an $\in$-chain, then we close this chain under intersections and show that the resulting set is still an $\in$-chain.

Claim 1.13. *The set $\mathcal{M}_q \cup \mathcal{M}_r$ is an $\in$-chain.*

Proof. Note that any model of $\mathcal{M}_r \setminus M$ is either in $[M, H(\aleph_2))_r$ or belongs to an interval of the form $[N\cap M, N)_r$, for some $N \in \pi_1(r \restriction M)$. Consider one such interval $[N\cap M, N)_r$. Since $N \in r \restriction M$ and $q \leq r \restriction M$ we have that $N \in \mathcal{M}_q$. The models in $\mathcal{M}_r \cap [N\cap M, N)_r$ are an $\in$-chain. The least model on this chain is $N \cap M$ and the last one belongs to N. Consider the $\in^*$-largest model P of $\mathcal{M}_q$ below N. Since $q \in M$ we have that $P \in M$. Moreover, since $\mathcal{M}_q$ is an $\in$-chain we have that also $P \in N$, therefore $P \in N \cap M$. Similarly, the least model of $\mathcal{M}_r$ in $[M, H(\aleph_2))_r$ is M and it contains the top model of $\mathcal{M}_q$. Therefore, $\mathcal{M}_q \cup \mathcal{M}_r$ is an $\in$-chain. $\square$

We now close $\mathcal{M}_q \cup \mathcal{M}_r$ under intersections and check that it is still an $\in$-chain. We let $Q \in \mathcal{M}_q \setminus \mathcal{M}_r$ and consider models of the form $Q \cap R$, for $R \in \mathcal{M}_r$.

Case 1: $Q \in \pi_0(q)$. We show by $\in^*$-induction on R that $Q \cap R$ is already on the chain $\mathcal{M}_q$. Since $Q \in M$ and Q is countable we have that $Q \subseteq M$. Therefore, $Q \cap R = Q \cap (R \cap M)$. We know that $R, M \in \mathcal{M}_r$ and $\mathcal{M}_r$ is closed under intersections, so $R \cap M \in \mathcal{M}_r$. By replacing R by $R \cap M$ we may assume that R is countable and below M in $\mathcal{M}_r$. If $R \in M$ then $R \in \mathcal{M}_q$ and $\mathcal{M}_q$ is closed under intersection, so $Q \cap R \in \mathcal{M}_q$. If $R \in \mathcal{M}_r \setminus M$ then it belongs to an interval of the form $[N \cap M, N)_r$, for some $N \in \pi_1(r \restriction M)$. Since N is uncountable and $R \in^* N$ it follows that $R \subseteq N$. If there is no uncountable model in the interval $[N \cap M, R)_r$ then we have that $N \cap M \subseteq R \subseteq N$. It follows that

$$Q \cap (N \cap M) \subseteq Q \cap R \subseteq Q \cap N.$$

However, Q is a subset of M and so $Q \cap (N \cap M) = Q \cap N$. Therefore, $Q \cap R = Q \cap N$ and since $Q, N \in \mathcal{M}_q$ we have again that $Q \cap N \in \mathcal{M}_q$. Now, suppose there is an uncountable model in $[N \cap M, R)_r$ and let S be the $\in^*$-largest such model. Since all the models in the interval $(S, R)_r$ are countable we have that $S \in R$. On the other hand, S is uncountable and above $N \cap M$ in $\mathcal{M}_r$. It follows that $N \cap M \subseteq S$. Now, consider the model $R^* = R \cap S$. It is below S in $\mathcal{M}_r$. We claim that $Q \cap R = Q \cap R^*$. To see this note that, since $Q \subseteq M$ and $R \subseteq N$, we have

$$Q \cap R \subseteq Q \cap (N \cap M) \subseteq Q \cap S.$$

Therefore, $Q \cap R^* = Q \cap (R \cap S) = Q \cap R$. Since R^* is below R in $\mathcal{M}_r$, by the inductive assumption, we have that $Q \cap R^* \in \mathcal{M}_q$.

Case 2: $Q \in \pi_1(q)$. We first show that the largest element of $\mathcal{M}_q \cup \mathcal{M}_r$ below Q is in $\mathcal{M}_q$. To see this note that by Fact 1.7 any model, say S, in $\mathcal{M}_r \setminus M$ which is below M under $\in^*$ belongs to an interval of the form $[N \cap M, N)_r$, for some $N \in \pi_1(r \restriction M)$. By our assumption, $Q \in \mathcal{M}_q \setminus \mathcal{M}_r$ so N is distinct from Q. Since $N, Q \in \mathcal{M}_q$ and they are both uncountable it follows that either $Q \in N$ or $N \in Q$. In the first case, $Q \in N \cap M$, i.e. Q is $\in^*$-below S. In the second case, $S \in^* N \in^* Q$ and $N \in M$.

We now consider models of the form $Q \cap R$, for $R \in \mathcal{M}_r$. If R is uncountable then either $Q \subseteq R$ or $R \subseteq Q$ so $Q \cap R$ is in $\mathcal{M}_q \cup \mathcal{M}_r$. If R is countable and below Q on the chain $\mathcal{M}_q \cup \mathcal{M}_r$ then $R \subseteq Q$, so $Q \cap R = R$. If $R \in \mathcal{M}_r \cap M$ then $R \in \mathcal{M}_q$ and since $\mathcal{M}_q$ is closed under intersections we have that $Q \cap R \in \mathcal{M}_q$. So, suppose $R \in \pi_0(r) \setminus M$. By Fact 1.7 we know that R is either in $[M, H(\aleph_2))_r$ or in $[N \cap M, N)_r$, for some $N \in \pi_1(r \restriction M)$. We show by $\in^*$-induction that $Q \cap R$ is either in $\mathcal{M}_q \cup \mathcal{M}_r$ or is equal to

Q_{δ_R} and moreover $\delta_R \geq \delta_M$. Consider the case $R \in [M, H(\aleph_2))_r$. If there is no uncountable S in the interval $(M, R)_r$ then $M \subseteq R$. Therefore, $Q \in R$ and $\delta_R \geq \delta_M$. Since $Q \in R$ then $Q \cap R = Q_{\delta_R}$. If there is an uncountable model in the interval $(M, R)_r$ let S be the largest such model. Since Q is below S in the $\mathcal{M}_q \cup \mathcal{M}_r$ chain we have $Q \subseteq S$, so if we let $R^* = R \cap S$, then $Q \cap R^* = Q \cap R$, and moreover $\delta_{R^*} = \delta_R$. Therefore, we can use the inductive hypothesis for R^*. The case when R belongs to an interval of the form $[N \cap M, N)_r$, for some $N \in \pi_1(r \restriction M) \cup \{H(\aleph_2)\}$ is treated in the same way.

The upshot of all of this is that when we close $\mathcal{M}_q \cup \mathcal{M}_r$ under intersections the only new models we add are of the form Q_ξ, for $Q \in \pi_1(\mathcal{M}_q \setminus \mathcal{M}_r)$, and finitely many countable ordinals $\xi \geq \delta_M$. These models form an $\in$-chain, say C_Q. In particular, the case $R = M$ falls under the last case of the previous paragraph, therefore $Q_{\delta_M} = Q \cap M$ is the $\in^*$-least member of C_Q. Moreover, if Q' is the predecessor of Q in $\mathcal{M}_q \cup \mathcal{M}_r$, then Q' belongs to both Q and M and hence it belongs to Q_{δ_M}. The largest member of C_Q is a member of Q since it is of the form Q_ξ, for some countable ξ. Thus, adding all these chains to $\mathcal{M}_q \cup \mathcal{M}_r$ we preserve the fact that we have an $\in$-chain. $\square$

As an immediate consequence of Lemma 1.12 we have the following.

Theorem 1.14. $\mathbb{M}$ *is* $\mathcal{E}^2$*-strongly proper.*

Proof. Suppose $M \in \mathcal{E}^2$ and $p \in M \cap \mathbb{M}$. We shall show that p^M is $(M, \mathbb{M})$-strongly generic. To see this we for every condition $r \leq p^M$ we have to define a condition $r|M \in \mathbb{M} \cap M$ such that for every $q \in \mathbb{M} \cap M$ if $q \leq r|M$ then q and r are compatible. If we let $r|M$ simply be $r \cap M$ this is precisely the statement of Lemma 1.12. $\square$

Corollary 1.15. *The forcing* $\mathbb{M}$ *is proper and preserves* ω_2. $\square$

2. Adding a club in ω_2 with finite conditions

We now present a version of the Friedman-Mitchell (see [3] and [7]) forcing for adding a club to ω_2 with finite conditions. This will be achieved by adding a working part to the side conditions.

Definition 2.1. *Let* $\mathbb{M}_2$ *be the forcing notion whose elements are triples* $p = (F_p, A_p, \mathcal{M}_p)$*, where* $F_p \in [\omega_2]^{<\omega}$*,* A_p *is a finite collection of intervals of the form* $(\alpha, \beta]$*, for some* $\alpha, \beta < \omega_2$*,* $\mathcal{M}_p \in \mathbb{M}$*, and*

(1) $F_p \cap \bigcup A_p = \emptyset$,
(2) *if* $M \in \mathcal{M}_p$ *and* $I \in A_p$*, then either* $I \in M$ *or* $I \cap M = \varnothing$.

The order on $\mathbb{M}_2$ *is coordinatewise reverse inclusion, i.e.* $q \leq p$ *if* $F_p \subseteq F_q$*,* $A_p \subseteq A_q$ *and* $\mathcal{M}_p \subseteq \mathcal{M}_q$.

The information carried by a condition p is the following. The points of F_p are going to be in the generic club, and the intervals in A_p are a partial description of the complement of that club. The side conditions are there to ensure that the forcing is $\mathcal{E}^2$-strongly proper. It should be pointed our that a condition r may force some ordinals to be in the generic club even though they are not explicitly in F_r. The reason is that we may not be able to exclude them by intervals which satisfy conditions (1) and (2) of Definition 2.1.

Fact 2.2. *If $p \in \mathbb{M}_2$ and $M \in \mathcal{M}_p$ then* $\sup(M \cap \omega_2) \notin \bigcup A_p$.

Proof. Any interval I which contains $\sup(M \cap \omega_2)$ would have to intersect M without being an element of M. This contradicts condition (2) of Definition 2.1. □

Fact 2.3. *Suppose $p \in \mathbb{M}_2$, $M \in \mathcal{M}_p$ and $\gamma \in F_p$. Then*

$$\min(M \setminus \gamma), \sup(M \cap \gamma) \notin \bigcup A_p.$$

Proof. Suppose $\gamma \in F_p$ and let $I \in A_p$. Then I is of the form $(\alpha, \beta]$, for some ordinals $\alpha, \beta < \omega_2$. Since p is a condition we know that $\gamma \notin I$. By condition (2) of Definition 2.1 we know that either $I \cap M = \emptyset$ or $I \in M$. If $I \cap M = \emptyset$ then $\sup(M \cap \gamma), \min(M \setminus \gamma) \notin I$. Assume now that $I \in M$. Since $\gamma \notin I$ we have that either $\gamma \leq \alpha$ or $\gamma > \beta$. Suppose first that $\gamma \leq \alpha$. Since $\alpha \in M$ it follows $\min(M \setminus \gamma) \leq \alpha$ and so $\min(M \setminus \gamma) \notin I$. Clearly, also $\sup(M \cap \gamma) \notin I$. Suppose now $\gamma > \beta$. In that case, clearly, $\min(M \setminus \gamma) \notin I$. Also, since $\beta \in M$ it follows that $\beta < \sup(M \cap \gamma)$ and so $\sup(M \cap \gamma) \notin I$.

□

Definition 2.4. *Suppose $p \in \mathbb{M}_2$ and $M \in \mathcal{M}_p$. We say that p is M-*complete *if*

1. $\sup(N \cap \omega_2) \in F_p$, *for all* $N \in \mathcal{M}_p$,
2. $\min(M \setminus \gamma), \sup(M \cap \gamma) \in F_p$, *for all* $\gamma \in F_p$.

We say that p is complete *if it is M-complete, for all $M \in \mathcal{M}_p$.*

The following is straightforward.

Fact 2.5. *Suppose $p \in \mathbb{M}_2$ and $M \in \mathcal{M}_p$. Then there is an M-complete condition q which is equivalent to p. We call the least, under inclusion, such condition the M-*completion *of p.*

Proof. First let $F^* = F_p \cup \{\sup(N \cap \omega_2) : N \in \mathcal{M}_p\}$. Then let

$$F_q = F^* \cup \{\sup(M \cap \gamma) : \gamma \in F^*\} \cup \{\min(M \setminus \gamma) : \gamma \in F^*\}.$$

Let $A_q = A_p$ and $\mathcal{M}_q = \mathcal{M}_p$. It is straightforward to check that $q = (F_q, A_q, \mathcal{M}_q)$ is a condition equivalent to p and M-complete. □

Remark 2.6. Note that in the above fact q is M-complete for a single $M \in \mathcal{M}_p$. We may not be able find q which is complete, i.e. M-complete, for all $M \in \mathcal{M}_q$. To see this, suppose there are $M, N \in \mathcal{M}_p$ such that

$$\lim(M \cap N \cap \omega_2) \neq \lim(M \cap \omega_2) \cap \lim(N \cap \omega_2).$$

Note that if $\gamma \in M \cap N$ then either $M \cap \gamma \subseteq N$ or $N \cap \gamma \subseteq M$. Therefore, the least common limit of M and N which is not a limit of $M \cap N$ is above $\sup(M \cap N)$. If q is an extension of p which is complete then $\sup(M \cap N) \in F_q$, because $M \cap N \in \mathcal{M}_q$. Now, $\sup(M \cap N) \notin M \cap N$. Let us assume, for concreteness, that $\sup(M \cap N) \notin M$. We can define inductively a strictly increasing sequence $(\gamma_n)_n$ by setting $\gamma_0 = \sup(M \cap N)$ and

$$\gamma_{n+1} = \begin{cases} \min(M \setminus \gamma_n) & \text{if } n \text{ is even} \\ \min(N \setminus \gamma_n) & \text{if } n \text{ is odd.} \end{cases}$$

Since, q was assumed to be both M-complete and N-complete we would have that $\gamma_n \in F_q$, for all n. This means that F_q would have to be infinite, which is a contradiction. We do not know if such a pair of models can exist in a condition in $\mathbb{M}$. Nevertheless, we will later present a variation of $\mathbb{M}_2$ in which this situation does not occur and in which the set of fully complete conditions is dense.

We now come back to Lemma 1.8 and observe that it is valid also for $\mathbb{M}_2$.

Lemma 2.7. *Let $M \in \mathcal{E}^2$ and let $p \in \mathbb{M}_2 \cap M$. Then there is a new condition, which we will call p^M, that is the smallest element of $\mathbb{M}_2$ extending p such that $M \in \mathcal{M}_{p^M}$.*

Proof. If $M \in \mathcal{E}^2_1$ then simply let $p^M = (F_p, A_p, \mathcal{M}_p \cup \{M\})$. If $M \in \mathcal{E}^2_0$, then, as in Lemma 1.8, we let $\mathcal{M}_{p^M}$ be the closure of $\mathcal{M}_p \cup \{M\}$ under intersection. We also let $F_{p^M} = F_p$ and $A_{p^M} = A_p$. We need to check that conditions (1) and (2) of Definition 2.1 are satisfied for p^M, but this is straightforward. □

Our next goal is to show that $\mathbb{M}_2$ is $\mathcal{E}^2$-strongly proper. We first establish the following.

Lemma 2.8. *Suppose $p \in \mathbb{M}_2$ and $M \in \mathcal{M}_p$. Then p is $(M, \mathbb{M}_2)$-strongly generic.*

Proof. We need to define, for each $r \leq p$ a restriction $r|M \in M$ such that for every $q \in M$ if $q \leq r|M$ then q and r are compatible. So, suppose $r \leq p$. By replacing r with its M-completion we may assume that r is M-complete. We define

$$r|M = (F_r \cap M, A_r \cap M, \mathcal{M}_r \cap M).$$

By Facts 1.7 or 1.6 according to whether M is countable or not we have that $\mathcal{M}_r \cap M \in \mathbb{M}$ and therefore $r|M \in \mathbb{M}_2 \cap M$. We need to show that for every $q \in M$ if $q \leq r|M$ then q and r are compatible.

If $M \in \mathcal{E}_1^2$ we already know that $\mathcal{M}_s = \mathcal{M}_q \cup \mathcal{M}_r$ is an $\in$-chain closed under intersection. Let $F_s = F_q \cup F_r$ and $A_s = A_q \cup A_r$. Finally, let $s = (F_s, A_s, \mathcal{M}_s)$. It is straightforward to check that s is a condition and $s \leq r, q$.

We now concentrate on the case $M \in \mathcal{E}_0^2$. We define a condition s as follows. We let $F_s = F_q \cup F_r$, $A_s = A_q \cup A_r$ and

$$\mathcal{M}_s = \mathcal{M}_q \cup \mathcal{M}_r \cup \{Q \cap R : Q \in \mathcal{M}_q, R \in \mathcal{M}_r\}.$$

We need to check that $s \in \mathbb{M}_2$. By Lemma 1.13 we know that $\mathcal{M}_s$ is an $\in$-chain closed under intersection. Therefore we only need to check that (1) and (2) of Definition 2.1 are satisfied for s. First we check (1).

Claim 2.9. $F_s \cap \bigcup A_s = \emptyset$.

Proof. It suffices to check that $F_q \cap \bigcup A_r = \emptyset$ and $F_r \cap \bigcup A_q = \emptyset$. Suppose first $\gamma \in F_q$ and $I \in A_r$. Since $M \in \mathcal{M}_r$ we have, by (2) of Definition 2.1, that either $I \cap M = \emptyset$ or $I \in M$. If $I \cap M = \emptyset$ then, since $\gamma \in M$, we have that $\gamma \notin I$. If $I \in M$ then $I \in A_r \cap M$ and, since $q \leq r|M$, it follows that $I \in A_q$. Now, q is a condition, so $\gamma \notin I$.

Suppose now $\gamma \in F_r$ and $I \in A_q$. If $\gamma \in F_r \cap M$ then $\gamma \in F_q$. Therefore $\gamma \notin I$. Suppose now $\gamma \in F \setminus M$. Since r is M-complete $\gamma^* = \min(M \setminus \gamma) \in F_r$. Then $\gamma^* \in F_r \cap M$ and so $\gamma^* \in F_q$. Now, $I \in M$ and so if $\gamma \in I$ then $\gamma^* \in I$, which would be a contradiction. Therefore $\gamma \notin I$.

□

We now turn to condition (2) of Definition 2.1.

Claim 2.10. *If $Q \in \mathcal{M}_q$ and $I \in A_r$ then either $I \in Q$ or $I \cap Q = \emptyset$.*

Proof. Since $M \in \mathcal{M}_r$ we have that either $I \in M$ or $I \cap M = \emptyset$. If $I \in M$ then $I \in A_r \cap M$ and so $I \in A_q$. Since q is a condition we have that either $I \in Q$ or $I \cap Q = \emptyset$. So, suppose $I \cap M = \emptyset$. If $Q \in \mathcal{E}_0^2$ then $Q \subseteq M$ and so $Q \cap I = \emptyset$, as well. If $Q \in \mathcal{E}_1^2$ then $Q \cap \omega_2$ is an initial segment of ω_2, say γ. Now, if $I \cap Q \neq \emptyset$ and $I \notin Q$ we would have that $\gamma \in I$. Since $\gamma \in M$ this contradicts the fact that $I \cap M = \emptyset$. □

Claim 2.11. *If $R \in \mathcal{M}_r$ and $I \in A_q$ then either $I \in R$ or $I \cap R = \emptyset$.*

Proof. Assume first that $R \in \mathcal{E}_1^2$. Then $R \cap \omega_2$ is an initial segment of ω_2, say γ. If $I \cap R \neq \emptyset$ and $I \notin R$ then $\gamma \in I$. Now, since r is M-complete we have that $\gamma \in F_r$. If $\gamma \in M$ then $\gamma \in F_q$ and this would contradict the fact that q is a condition. If $\gamma \notin M$ let $\gamma^* = \min(M \setminus \gamma)$. Then, again by M-completeness of r, we have that $\gamma^* \in F_r$. However, $\gamma^* \in M$

and therefore $\gamma^* \in F_q$. Since $I \in A_q$ and $q \in M$ we have that $I \in M$. If $\gamma \in I$ we would also have that $\gamma^* \in I$, which contradicts the fact that q is a condition.

We now consider the case $R \in \mathcal{E}_0^2$. We will show by $\in^*$-induction on the chain $\mathcal{M}_r$ that either $I \cap R = \emptyset$ or $I \in R$. If $R \in M$ then $R \in \mathcal{M}_q$ so this is clear. If $R \notin M$ then R either belongs to $[M, H(\aleph_2))_r$ or else belongs to $[N \cap M, N)_r$, for some uncountable $N \in \mathcal{M}_r \cap M$.

Suppose $R \in [N \cap M, N)_r$, for some $N \in \pi_1(\mathcal{M}_r \cap M)$. Since $I \in A_q$ and $N \in \mathcal{M}_q$ we have that $I \in N$ or $I \cap N = \emptyset$. On the other hand, $R \subseteq N$ so if $I \cap N = \emptyset$ then also $I \cap R = \emptyset$. If $I \in N$ then, since $q \in M$ and $I \in A_q$, we have that $I \in M$ and so $I \in N \cap M$. If there are no uncountable models in the interval $[N \cap M, R)_r$ then $N \cap M \subseteq R$ and so $I \in R$. If there is an uncountable model in this interval let S be the largest such model. Now, $N \cap M \subseteq S$ and so $I \in S$ and $I \subseteq S$. It follows that if $I \cap R \neq \emptyset$ then also $I \cap R \cap S \neq \emptyset$. Let $R^* = R \cap S$. Then $R^* \in \mathcal{M}_r$ and R^* is below R in the $\in^*$-ordering. By the inductive assumption we would have that $I \in R^*$ and so $I \in R$. The case when $R \in [M, H(\aleph_2))_r$ is treated in the same way. □

Finally, suppose $Q \in \mathcal{M}_q$, $R \in \mathcal{M}_r$ and $I \in A_q \cup A_r$. Consider the relation between the model $Q \cap R$ and I. If I belongs to both Q and R then it belongs to $Q \cap R$. If I is disjoint from Q or R it is also disjoint from $Q \cap R$. This completes the proof that s is a condition. Since $s \leq q, r$ it follows that q and r are compatible.

□

Now, by Lemmas 2.7 and 2.8 we have the following.

Theorem 2.12. *The forcing $\mathbb{M}_2$ is $\mathcal{E}^2$-strongly proper. Hence it is proper and preserves ω_2.* □

Suppose now G is V-generic filter for the forcing notion $\mathbb{M}_2$. We can define

$$C_G = \bigcup\{F_p : p \in G\} \text{ and } U_G = \bigcup\bigcup\{A_p : p \in G\}.$$

Then $C_G \cap U_G = \emptyset$. Moreover, by genericity, $C_G \cup U_G = \omega_2$. Since U_G is a union of open intervals it is open in the order topology. Therefore, C_G closed and, again by genericity, it is unbounded in ω_2. Unfortunately, we cannot say much about the generic club C_G. For reasons explained in Remark 2.6, we cannot even say that it does not contain infinite subsets which are in the ground model. In order to circumvent this problem, we now define a variation of the forcing notion $\mathbb{M}_2$. We start by some definitions.

Definition 2.13. *Suppose $M, N \in \mathcal{E}^2$. We say that M and N are* lim-*compatible if*

$$\lim(M \cap N \cap \omega_2) = \lim(M \cap \omega_2) \cap \lim(N \cap \omega_2).$$

Remark 2.14. Clearly, this conditions is non trivial only if both M and N are countable. We will abuse notation and write $\lim(M)$ for $\lim(M \cap \omega_2)$.

We now define a version of the forcing notion $\mathbb{M}$.

Definition 2.15. *Let $\mathbb{M}^*$ be the suborder of $\mathbb{M}$ consisting of conditions $p = \mathcal{M}_p$ such that any two models in $\mathcal{M}_p$ are* lim-*compatible.*

We have the following version of Lemma 1.8.

Lemma 2.16. *Let $M \in \mathcal{E}^2$ and let $p \in \mathbb{M}^* \cap M$. Then there is a new condition, which we will call p^M, that is the smallest element of $\mathbb{M}^*$ extending p such that $M \in \mathcal{M}_{p^M}$.*

Proof. If $M \in \mathcal{E}^2_1$ then simply let $p^M = M_p \cup \{M\}$. If $M \in \mathcal{E}^2_0$, then we let $\mathcal{M}_{p^M}$ be the closure of $\mathcal{M}_p \cup \{M\}$ under intersection. Then, thanks to Lemma 1.8, we just need to check that the models in $\mathcal{M}_{p^M}$ are lim-compatible. Suppose $P \in \pi_0(p)$. Then $P \in M$ and hence $P \subseteq M$. Therefore, P and M are lim-compatible. Suppose now $P \in \pi_1(p)$. Then $P \cap \omega_2$ is an initial segment of ω_2, say γ. Therefore

$$\lim(M \cap P) = \lim(M \cap \gamma) = \lim(M) \cap (\gamma + 1) = \lim(M) \cap \lim(P),$$

and so P and M are lim-compatible. We also need to check that, for any $P, Q \in \mathcal{M}_p$, the models $P \cap M$ and $Q \cap M$, as well as $P \cap M$ and Q are lim-compatible, but this is straightforward. $\square$

We now have a version of Lemma 1.12.

Lemma 2.17. *Suppose $r \in \mathbb{M}^*$ and $M \in \mathcal{M}_r$. Let $q \in \mathbb{M}^* \cap M$ be such that $q \leq r \cap M$. Then q and r are compatible in $\mathbb{M}^*$.*

Proof. If M is uncountable then one can easily check that $\mathcal{M}_s = \mathcal{M}_q \cup \mathcal{M}_r$ is $\in$-chain closed under intersection and that any two models in $\mathcal{M}_s$ are lim-compatible.

Suppose now M is countable and let

$$\mathcal{M}_s = \mathcal{M}_q \cup \mathcal{M}_r \cup \{Q \cap R : Q \in \mathcal{M}_q, R \in \mathcal{M}_r\}.$$

Thanks to Lemma 1.12 we know that $\mathcal{M}_s$ is an $\in$-chain closed under intersection. It remains to check that any two models in $\mathcal{M}_s$ are lim-compatible.

Claim 2.18. *If $Q \in \pi_0(\mathcal{M}_q)$ and $R \in \pi_0(\mathcal{M}_r)$, then Q and R are lim-compatible.*

Proof. We show this by $\in^*$-induction on R. Since $Q \in \mathcal{M}_q$ then $Q \in M$ and, since Q is countable, we have that $\lim(Q) \subseteq M$. Moreover, since R and M are both in $\mathcal{M}_r$, we have that $\lim(R \cap M) = \lim(R) \cap \lim(M)$, and so

$$\lim(Q) \cap \lim(R) = \lim(Q) \cap \lim(R) \cap \lim(M) = \lim(Q) \cap \lim(R \cap M).$$

Hence, without loss of generality we can assume R to be $\in^*$-below M. If $R \in M$ then $R \in \mathcal{M}_q$ and so Q and R are lim-compatible. Assume now, $R \notin M$. Then by Fact 1.7 there is $N \in \pi_1(\mathcal{M}_r \cap M)$ such that $R \in [N \cap M, N)_r$. We may also assume Q is $\in^*$-below N, otherwise we could replace Q by $Q \cap N$. Hence $Q \subseteq N \cap M$. If there are no uncountable model in the interval $[N \cap M, R)_r$, then $N \cap M \subseteq R$ and since $Q \in N \cap M$ we have $Q \in R$. Therefore, Q and R are lim-compatible. Otherwise, let S be the $\in^*$-largest uncountable model in $[N \cap M, R)_r$. Then $Q \in S$ and $S \cap \omega_2$ is an initial segment of ω_2. Let $R^* = R \cap S$. It follows that $\lim(R) \cap \lim(Q) = \lim(R^*) \cap \lim(Q)$. By the inductive assumption we have that $\lim(R^*) \cap \lim(Q) = \lim(R^* \cap Q)$ and hence $\lim(R) \cap \lim(Q) = \lim(R \cap Q)$. $\square$

Now, we need to check that any two models in $\mathcal{M}_s$ are lim-compatible. So, suppose $S, S^* \in \mathcal{M}_s$. We may assume S and S^* are both countable and of the form $S = Q \cap R$, $S^* = Q^* \cap R^*$, for $Q, Q^* \in \mathcal{M}_q$ and $R, R^* \in \mathcal{M}_r$. Then

$$\lim((Q \cap R) \cap (Q^* \cap R^*)) = \lim((Q \cap Q^*) \cap (R \cap R^*))$$

and by Claim 2.18

$$\lim((Q \cap Q^*) \cap (R \cap R^*)) = \lim(Q \cap Q^*) \cap \lim(R \cap R^*),$$

because $Q \cap Q^* \in \mathcal{M}_q$ and $R \cap R^* \in \mathcal{M}_r$. Moreover, we have $= \lim(Q \cap Q^*) = \lim(Q) \cap \lim(Q^*)$ and $\lim(R \cap R^*) = \lim(R) \cap \lim(R^*)$, since the elements of $\mathcal{M}_q$, respectively $\mathcal{M}_r$, are lim-compatible. Finally, again by Claim 2.18, we have

$$\lim(Q) \cap \lim(R) \cap \lim(Q^*) \cap \lim(R^*) = \lim(Q \cap R) \cap \lim(Q^* \cap R^*).$$

$\square$

We now define a variation of the forcing $\mathbb{M}_2$ which will have some additional properties.

Definition 2.19. *Let $\mathbb{M}_2^*$ be the forcing notion whose elements are triples $p = (F_p, A_p, \mathcal{M}_p)$, where $F_p \in [\omega_2]^{<\omega}$, A_p is a finite collection of intervals of the form $(\alpha, \beta]$, for some $\alpha, \beta < \omega_2$, $\mathcal{M}_p \in \mathbb{M}^*$, and*

(1) $F_p \cap \bigcup A_p = \emptyset$,
(2) *if $M \in \mathcal{M}_p$ and $I \in A_p$, then either $I \in M$ or $I \cap M = \varnothing$,*

The order on $\mathbb{M}_2^*$ is coordinatewise reverse inclusion, i.e. $q \leq p$ if $F_p \subseteq F_q$, $A_p \subseteq A_q$ and $\mathcal{M}_p \subseteq \mathcal{M}_q$.

Remark 2.20. Note that the only difference between $\mathbb{M}_2^*$ and $\mathbb{M}_2$ is that for p to be in $\mathbb{M}_2^*$ we require that $\mathcal{M}_p \in \mathbb{M}^*$, i.e. the models in $\mathcal{M}_p$ are pairwise lim-compatible.

We can now use Lemmas 2.18 and 2.17 to prove the analogs of Lemmas 2.7 and 2.8 for $\mathbb{M}_2^*$. We then obtain the following.

Theorem 2.21. *The forcing notions $\mathbb{M}_2^*$ is $\mathcal{E}^2$-strongly proper. Hence, it is proper and preserves ω_2.* □

Let G^* is V-generic filter for $\mathbb{M}_2^*$. As in the case of the forcing $\mathbb{M}_2$, we define

$$C_G^* = \bigcup\{F_p : p \in G^*\} \text{ and } U_G^* = \bigcup\bigcup\{A_p : p \in G^*\}.$$

As before C_G^* is forced to be a club in ω_2. Our goal now is to show that it does not contain any infinite subset from the ground model. For this we will need the following lemma which explains the reason for the requirement of lim-compatibility for models $\mathcal{M}_p$, for conditions p in $\mathbb{M}_2^*$.

Lemma 2.22. *The set of complete conditions is dense in $\mathbb{M}_2^*$.*

Proof. Consider a condition $p \in \mathbb{M}_2^*$. For each $M \in \mathcal{M}_p$ we consider functions $\mu_M, \sigma_M : \omega_2 \to \omega_2$ defined as follows:

$$\mu_M(\alpha) = \min(M \setminus \alpha) \text{ and } \sigma_M(\alpha) = \sup(M \cap \alpha).$$

To obtain a complete condition extending p we first define:

$$F_p^* = F_p \cup \{\sup(M \cap \omega_2) : M \in \mathcal{M}_p\}.$$

We then let $\bar{F}_p$ be the closure of F_p^* under the functions μ_M and σ_M, for $M \in \mathcal{M}_p$. Then $q = (\bar{F}_p, A_p, \mathcal{M}_p)$ will be the required complete condition extending p. The main point is to show the following.

Claim 2.23. *$\bar{F}_p$ is finite.*

Proof. Let $L = \bigcup\{\lim(M) : M \in \mathcal{M}_p\}$. For each $\gamma \in L$ let

$$Y(p, \gamma) = \{M \in \mathcal{M}_p : \gamma \in \lim(M)\}.$$

and let $M(p, \gamma) = \bigcap Y(p, \gamma)$. Then, since $\mathcal{M}_p$ is closed under intersection $M(p, \gamma) \in \mathcal{M}_p$. Since the models in $\mathcal{M}_p$ are lim-compatible it follows that $\gamma \in \lim(M(p, \gamma))$. Thus, $M(p, \gamma)$ is the least (under inclusion) model in $\mathcal{M}_p$ which has γ as its limit point. For each $\gamma \in L$ pick an ordinal $f(\gamma) \in$

$M(p,\gamma)\cap\gamma$ above $\sup(F_p^*\cap\gamma)$ and $\sup(M\cap\gamma)$, for all $M\in\mathcal{M}_p\setminus Y(p,\gamma)$. For a limit $\gamma\in\omega_2\setminus L$ let

$$f(\gamma)=\sup\{\sup(M\cap\gamma):M\in\mathcal{M}_p\}.$$

Notice now that for any limit γ and any $M\in\mathcal{M}_p$, if $\xi\notin(f(\gamma),\gamma)$ then $\mu_M(\xi),\sigma_M(\xi)\notin(f(\gamma),\gamma)$. Since $\bar{F}_p$ is the closure of F_p^* under the functions μ_M and σ_M, for $M\in\mathcal{M}_p$, and $F_p^*\cap(f(\gamma),\gamma)=\emptyset$, for all limit γ, it follows that $\bar{F}_p\cap(f(\gamma),\gamma)=\emptyset$, for all limit γ. This means that $\bar{F}_p$ has no limit points and therefore is finite. □

□

Lemma 2.24. *Let $p\in\mathbb{M}_2^*$ be a complete condition, and let $\gamma\in\omega_2\setminus F_p$. Then there is a condition $q\le p$ such that $\gamma\in I$, for some $I\in A_q$.*

Proof. Without loss of generality we can assume that there is an $M\in\mathcal{M}_p$ such that $\sup(M\cap\omega_2)>\gamma$, otherwise we could let

$$q=(F_p,A_p\cup\{(\eta,\gamma]\},\mathcal{M}_p),$$

for some $\eta<\gamma$ sufficiently large so that $(\eta,\gamma]$ does not intersect any model in $\mathcal{M}_p$.

Now, since $\sup(M\cap\omega_2)\in F_p$, for every $M\in\mathcal{M}_p$, the set $F_p\setminus\gamma$ is nonempty. Let τ be $\min(F_p\setminus\gamma)$. Notice that for every model $M\in\mathcal{M}_p$ either $\sup(M\cap\tau)<\gamma$, or $\tau\in\lim(M)$, because

$$\gamma<\sup(M\cap\tau)<\tau,$$

would contradict the minimality of τ. Moreover, if $\sup(M\cap\tau)=\gamma$, then γ would be in F_p, contrary to the hypothesis of the lemma.

Let

$$Y=\{M\in\mathcal{M}_p:\tau\in\lim(M)\}.$$

Without loss of generality we can assume $Y\neq\emptyset$, because otherwise we can let

$$q=(F_p,A_p\cup\{(\eta,\gamma]\},\mathcal{M}_p)$$

for some η sufficiently large so that $(\eta,\gamma]$ avoids $\sup(M\cap\tau)$, for every $M\in\mathcal{M}_p$. Let $M_0=\bigcap Y$. Since $\mathcal{M}_p$ is closed under intersection $M_0\in\mathcal{M}_p$. Moreover, since any two models in $\mathcal{M}_p$ are lim-compatible we have that $\tau\in\lim M_0$. Thus, M_0 is itself in Y and is contained in any member of Y. Therefore, if an interval I belongs to M_0, then it belongs to every model in Y. Let $\eta=\min(M_0\setminus\gamma)$. Since $\tau\in\lim(M_0)$ we have $\gamma\le\eta<\tau$. Since τ is the least element of F_p above γ it follows that $\eta\notin F_p$.

Claim 2.25. $\sup(M_0\cap\gamma)>\sup(F_p\cap\gamma)$.

Proof. Suppose ξ is an element of $F_p \cap \gamma$. Since p is M_0-complete, we also have $\min(M_0 \setminus \xi) \in F_p$. Notice that $\min(M_0 \setminus \xi) \neq \eta$, since $\eta \notin F_p$. Then

$$\xi \leq \min(M_0 \setminus \xi) < \gamma,$$

and so $\sup(M_0 \cap \gamma) > \xi$. □

Consider now some $M \in \mathcal{M}_p \setminus Y$. Then $\tau \notin \lim(M)$ and, since p is M-complete, we have that $\sup(M \cap \tau) \in F_p$. Since τ is the least element of F_p above γ it follows that $\sup(M \cap \tau) \in F_p \cap \gamma$. Now, pick an element $\eta' \in M_0$ above $\sup(F_p \cap \gamma)$ and let $I = (\eta', \eta]$. It follows that $I \in M$, for all $M \in Y$ and $I \cap M = \emptyset$, for all $M \in \mathcal{M}_p \setminus Y$. Therefore,

$$q = (F_p, A_p \cup \{I\}, \mathcal{M}_p)$$

is a condition stronger than p and $\gamma \in I$. Thus, q is as required. □

Corollary 2.26. *If G^* is a V-generic filter over $\mathbb{M}_2^*$, then the generic club C_G^* does not contain any infinite subset which is in V.* □

3. Strong chains of uncountable functions

We now consider the partial order $(\omega_1^{\omega_1}, <_{\text{fin}})$ of all functions from ω_1 to ω_1 ordered by $f <_{\text{fin}} g$ iff $\{\xi : f(\xi) \geq g(\xi)\}$ is finite. In [5] Koszmider constructed a forcing notion which preserves cardinals and adds an ω_2 chain in $(\omega_1^{\omega_1}, <_{\text{fin}})$. The construction uses an $(\omega_1, 1)$-morass which is a stationary coding set and is quite involved. In this section we present a streamlined version of this forcing which uses generalizes side conditions and is based on the presentation of Mitchell [6]. Before that we show that Chang's conjecture implies that there is no such chain. The argument is inspired by a proof of Shelah from [10]. A similar argument appears in [4].

Proposition 3.1. *Assume Chang's conjecture. Then there is no chain in $(\omega_1^{\omega_1}, <_{\text{fin}})$ of length ω_2.*

Proof. Assume towards contradiction that Chang's conjecture holds and $\{f_\alpha : \alpha < \omega_2\}$ is a chain in $(\omega_1^{\omega_1}, <_{\text{fin}})$. Given a function $g : I \to \omega_1$ and $\eta < \omega_1$ we let $\min(g, \eta)$ be the function defined by:

$$\min(g, \eta)(\zeta) = \min(g(\zeta), \eta).$$

For each $\alpha < \omega_2$, and $\xi, \eta < \omega_1$ we define a function $f_\alpha^{\xi,\eta}$ by:

$$f_\alpha^{\xi,\eta} = \min(f_\alpha \restriction [\xi, \xi + \omega), \eta).$$

Given $\xi, \eta < \omega_1$, the sequence $\{f_\alpha^{\xi,\eta} : \alpha < \omega_2\}$ is $\leq_{\text{fin}}$-increasing. We define a club $C^{\xi,\eta} \subseteq \omega_2$ as follows.

Case 1: If the sequence $\{f_\alpha^{\xi,\eta} : \alpha < \omega_2\}$ eventually stabilizes under $=_{\text{fin}}$ we let $C^{\xi,\eta} = \omega_2 \setminus \mu$, where μ is least such that $f_\nu^{\xi,\eta} =_{\text{fin}} f_\mu^{\xi,\eta}$, for all $\nu \geq \mu$.

Case 2: If the sequence $\{f^{\xi,\eta}_\alpha : \alpha < \omega_2\}$ does not stabilize we let $C^{\xi,\eta}$ be a club in ω_2 such that $f^{\xi,\eta}_\alpha \lneqq_{\text{fin}} f^{\xi,\eta}_\beta$, for all $\alpha, \beta \in C^{\xi,\eta}$ with $\alpha < \beta$. This means that for every such α and β the set

$$\{n : f_\alpha(\xi + n) < f_\beta(\xi + n) \leq \eta\}$$

is infinite.

Let $C = \bigcap\{C^{\xi,\eta} : \xi, \eta < \omega_1\}$. Then C is a club in ω_2. We define a coloring $c : [C]^2 \to \omega_1$ by

$$c\{\alpha, \beta\}_< = \max\{\xi : f_\alpha(\xi) \geq f_\beta(\xi)\}.$$

By Chang's conjecture we can find an increasing ω_1 sequence $S = \{\alpha_\rho : \rho < \omega_1\}$ of elements of C such that $c[[S]^2]$ is bounded in ω_1. Let $\xi = \sup(c[[S]^2]) + 1$. Therefore for every $\rho < \tau < \omega_1$ we have

$$f_{\alpha_\rho} \restriction [\xi, \omega_1) < f_{\alpha_\tau} \restriction [\xi, \omega_1).$$

Now, let $\eta = \sup(\text{ran}(f_{\alpha_1}\restriction[\xi, \xi + \omega)))$. It follows that for every n:

$$f_{\alpha_0}(\xi + n) < f_{\alpha_1}(\xi + n) \leq \eta.$$

Since $\alpha_0, \alpha_1 \in C^{\xi,\eta}$ it follows that $C^{\xi,\eta}$ was defined using Case 2. Therefore the sequence $\{f^{\xi,\eta}_{\alpha_\rho} : \rho < \omega_1\}$ is $\leq$-increasing and $f_{\alpha_\rho} \neq_{\text{fin}} f_{\alpha_\tau}$, for all $\rho < \tau$. For each $\rho < \omega_1$ let n_ρ be the least such that $f^{\xi,\eta}_{\alpha_\rho}(\xi + n_\rho) < f^{\xi,\eta}_{\alpha_{\rho+1}}(\xi + n_\rho)$. Then there is a integer n such that $X = \{\rho < \omega_1 : n_\rho = n\}$ is uncountable. It follows that the sequence $\{f_{\alpha_\rho}(\xi + n) : \rho \in X\}$ is strictly increasing. On the other hand it is included in η which is countable, a contradiction.

$\square$

Therefore, in order to add a strong ω_2-chain in $(\omega_1^{\omega_1}, <_{\text{fin}})$ we need to assume that Chang's conjecture does not hold. In fact, we will assume that there is an increasing function $g : \omega_1 \to \omega_1$ such that

(1) $g(\xi)$ is indecomposable, for all $\xi < \omega_1$,
(2) $\text{o.t.}(M \cap \omega_2) < g(\delta_M)$, for all $M \in \mathcal{E}^2_0$.

It is easy to add such a function by a preliminary forcing. For instance, we can add by countable conditions an increasing function g which dominates all the canonical functions c_α, for $\alpha < \omega_2$, and such that $g(\xi)$ is indecomposable, for all ξ. Moreover, we may assume that g is definable in the structure $(H(\aleph_2), \in, \trianglelefteq)$ and so it belongs to M, for all $M \in \mathcal{E}^2$.

Our plan is to add an ω_2-chain $\{f_\alpha : \alpha < \omega_2\}$ in $(\omega_1^{\omega_1}, <_{\text{fin}})$ below this function g. We can view this chain as a single function $f : \omega_2 \times \omega_1 \to \omega_1$. We want to use conditions of the form $p = (f_p, \mathcal{M}_p)$, where $f_p : A_p \times F_p \to \omega_1$ for some finite $A_p \subseteq \omega_2$ and $F_p \subseteq \omega_1$, and $\mathcal{M}_p \in \mathbb{M}$ is a side condition. Suppose $\alpha, \beta \in A_p$ with $\alpha < \beta$, and $M \in \pi_0(\mathcal{M}_p)$. Then M should localize the disagreement of f_α and f_β, i.e. p should force that the finite set $\{\xi : f_\alpha(\xi) \geq f_\beta(\xi)\}$ is contained in M. This means that if $\xi \in \omega_1 \setminus M$ then p makes the commitment that $f_\alpha(\xi) < f_\beta(\xi)$. Moreover, for every $\eta \in (\alpha, \beta) \cap M$ we should have that $f_\alpha(\xi) < f_\eta(\xi) < f_\beta(\xi)$. Therefore, p imposes that $f_\beta(\xi) \geq f_\alpha(\xi) + \text{o.t.}([\alpha, \beta) \cap M)$. This motivates the definition of the distance function below. Before defining the distance function we need to prove some general properties of side conditions. For a set of ordinals X we let $\overline{X}$ denote the closure of X in the order topology.

Fact 3.2. *Suppose $P, Q \in \mathcal{E}_0^2$ and $\delta_P \leq \delta_Q$.*

(1) *If $\gamma \in P \cap Q \cap \omega_2$ then $P \cap \gamma \subseteq Q \cap \gamma$.*
(2) *If P and Q are* lim-*compatible and $\gamma \in \overline{P \cap \omega_2} \cap \overline{Q \cap \omega_2}$ then $P \cap \gamma \subseteq Q \cap \gamma$.*

Proof. (1) For each $\alpha < \omega_2$ let e_α be the $\trianglelefteq$-least injection from α to ω_1. Then $P \cap \gamma = e_\gamma^{-1}[\delta_P]$ and $Q \cap \gamma = e_\gamma^{-1}[\delta_Q]$. Since $\delta_P \leq \delta_Q$ we have that $P \cap \gamma \subseteq Q \cap \gamma$.
(2) If $\gamma \in P \cap Q$ this is (1). Suppose γ is a limit point of either P or Q then it is also the limit point of the other. Since P and Q are lim-compatible we have that $\gamma \in \lim(P \cap Q)$. Then $P \cap \gamma = \bigcup\{e_\alpha^{-1}[\delta_P] : \alpha \in P \cap Q\}$ and $Q \cap \gamma = \bigcup\{e_\alpha^{-1}[\delta_Q] : \alpha \in P \cap Q\}$ Since $\delta_P \leq \delta_Q$ we conclude that $P \cap \gamma \subseteq Q \cap \gamma$. $\square$

Fact 3.3. *Suppose $p \in \mathbb{M}$ and $P, Q \in \pi_0(\mathcal{M}_p)$. If $\delta_P < \delta_Q$ and $P \subseteq Q$ then $P \in Q$.*

Proof. If there is no uncountable model in the interval $(P, Q)_p$, then $P \in Q$ by transitivity. Otherwise, let S be the $\in^*$-largest uncountable model below Q and we proceed by $\in^*$-induction. First note that $S \in Q$ by transitivity and if we let $Q^* = Q \cap S$ then $\delta_{Q^*} = \delta_Q$. Since $P \subseteq S$, we have that $P \subseteq Q^*$ and so Q^* is $\in^*$-above P. By the inductive assumption we have $P \in Q^* \subseteq Q$, as desired. $\square$

Definition 3.4. *Let $p = \mathcal{M}_p \in \mathbb{M}^*$, $\alpha, \beta \in \omega_2$ and let ξ be a countable ordinal. Then the binary relation $L_{p,\xi}(\alpha, \beta)$ holds if there is a $P \in \mathcal{M}_p$, with $\delta_P \leq \xi$, such that $\alpha, \beta \in \overline{P \cap \omega_2}$. In this case we will say that α and β are p, ξ-*linked.

Definition 3.5. *Let $p = \mathcal{M}_p \in \mathbb{M}^*$ and $\xi < \omega_1$. We let $C_{p,\xi}$ be the transitive closure of the relation $L_{p,\xi}$. If $C_{p,\xi}(\alpha, \beta)$ holds we say that α and β are p, ξ-*connected*. If $\alpha < \beta$ and α and β are p, ξ-connected we write $\alpha <_{p,\xi} \beta$.*

From Fact 3.2(2) we now have the following.

Fact 3.6. *Suppose $p = \mathcal{M}_p \in \mathbb{M}^*$ and $\xi < \omega_1$.*

(1) *Suppose* $\alpha < \beta < \gamma$ *are ordinal in* ω_2. *If* $L_{p,\xi}(\alpha,\gamma)$ *and* $L_{p,\xi}(\beta,\gamma)$ *hold, then so does* $L_{p,\xi}(\alpha,\beta)$.
(2) *If* $\alpha <_{p,\xi} \beta$ *then there is a sequence* $\alpha = \gamma_0 < \gamma_1 < \ldots < \gamma_n = \beta$ *such that* $L_{p,\xi}(\gamma_i,\gamma_{i+1})$ *holds, for all* $i < n$.

□

We now present some properties of the relation $<_{p,\xi}$, in order to define the distance function we will use in the definition of the main forcing.

Fact 3.7. *Let* $p = \mathcal{M}_p \in \mathbb{M}^*$ *and* $\xi < \omega_1$. *Suppose* $\alpha < \beta < \gamma < \omega_2$. *Then*
(1) *if* $\alpha <_{p,\xi} \beta$ *and* $\beta <_{p,\xi} \gamma$, *then* $\alpha <_{p,\xi} \gamma$,
(2) *if* $\alpha <_{p,\xi} \gamma$ *and* $\beta <_{p,\xi} \gamma$, *then* $\alpha <_{p,\xi} \beta$.

Proof. Part (1) follows directly from the definition of the relation $<_{p,\xi}$. To prove (2) let $\alpha = \gamma_0 < \ldots < \gamma_n = \gamma$ witness the p,ξ-connection between α and γ and let $\beta = \delta_0 < \ldots < \delta_l = \gamma$ witness the p,ξ-connection between β and γ. We have that $L_{p,\xi}(\gamma_i,\gamma_{i+1})$ holds, for all $i < n$, and $L_{p,\xi}(\delta_j,\delta_{j+1})$ holds, for all $j < l$. We prove that α and β are p,ξ-connected by induction on $n+l$. If $n = l = 1$ this is simply Fact 3.6(1). Let now $n,l > 1$. Assume for concreteness that $\delta_{l-1} \leq \gamma_{n-1}$. By Fact 3.6(1) $L_{p,\xi}(\delta_{l-1},\gamma_{n-1})$ holds; so $\alpha <_{p,\xi} \gamma_{n-1}$ and $\beta <_{p,\xi} \gamma_{n-1}$. Now, by the inductive assumption we conclude that α and β are p,ξ-connected, i.e. $\alpha <_{p,\xi} \beta$. The case $\gamma_{n-1} < \delta_{l-1}$ is treated similarly. □

The above lemma shows in (1) that the relation $<_{p,\xi}$ is transitive and in (2) that the set $(\omega_2, <_{p,\xi})$ has a tree structure. Since for every $M \in \mathcal{E}^2_0$ if $\delta_M \leq \xi$ then $\text{o.t.}(M \cap \omega_2) < g(\xi)$ and $g(\xi)$ is indecomposable we conclude that the height of $(\omega_2, <_{p,\xi})$ is at most $g(\xi)$. For every $\alpha <_{p,\xi} \beta$ we let $(\alpha,\beta)_{p,\xi} = \{\eta : \alpha <_{p,\xi} \eta <_{p,\xi} \beta\}$. We define similarly $[\alpha,\beta)_{p,\xi}$ and $(\alpha,\beta]_{p,\xi}$ and $[\alpha,\beta]_{p,\xi}$. If $0 <_{p,\xi} \beta$, i.e. β belongs to some $M \in \mathcal{M}_p$ with $\delta_M \leq \xi$ we write $(\beta)_{p,\xi}$ for the interval $[0,\beta)_{p,\xi}$. Thus, $(\beta)_{p,\xi}$ is simply the set of predecessors of β in $<_{p,\xi}$. If β does not belong to $\overline{M \cap \omega_2}$ for any $M \in \mathcal{M}_p$ with $\delta_M \leq \xi$ we leave $(\beta)_{p,\xi}$ undefined. Note that when defined $(\beta)_{p,\xi}$ is a closed subset of β in the ordinal topology.

Fact 3.8. *Let* $p \in \mathbb{M}^*$, $M \in \mathcal{M}_p$, $\xi \in M \cap \omega_1$ *and* $\beta \in M \cap \omega_2$. *Then* $(\beta)_{p,\xi} \subseteq M$. *Moreover, if we let* $p^* = p \cap M$ *then* $(\beta)_{p,\xi} = (\beta)_{p^*,\xi}$.

Proof. Let $\alpha <_{p,\xi} \beta$ and fix a sequence $\alpha = \gamma_0 < \gamma_1 < \ldots < \gamma_n = \beta$ such that $L_{p,\xi}(\gamma_i,\gamma_{i+1})$ holds, for all $i < n$. We proceed by induction on n. Suppose first $n = 1$ and let P witness that α and β are p,ξ-linked. Since $\delta_P < \delta_M$ we have by Fact 3.2 that $P \cap \beta \subseteq M$ and by Fact 3.3 that $P \cap M \in M$. Therefore $\alpha, \beta \in \overline{P \cap M \cap \omega_2} \subseteq M$ and so $P \cap M$ witnesses that α and β are p^*,ξ-linked. Consider now the case $n > 1$. By the same argument as in the case $n = 1$ we know that γ_{n-1} and β are

p^*, ξ-linked and then by the inductive hypothesis we conclude that α and β are p^*, ξ-connected. $\square$

Fact 3.9. *Let $p \in \mathbb{M}^*$, $M \in \mathcal{M}_p$, $\beta \in \omega_2 \setminus M$ and $\xi \in M \cap \omega_1$. If $(\beta)_{p,\xi} \cap M$ is non empty then it has a largest element, say η. Moreover, there is $Q \in \mathcal{M}_p \setminus M$ with $\delta_Q \leq \xi$ such that $\eta = \sup(Q \cap M \cap \omega_2)$.*

Proof. Assume $(\beta)_{p,\xi} \cap M$ is non empty and let η be its supremum. Note that η is a limit ordinal. Since $(\beta)_{p,\xi}$ is a closed subset of β in the order topology we know that either $\eta <_{p,\xi} \beta$ or $\eta = \beta$. By Fact 3.8 $(\beta)_{p,\xi} \cap M = (\beta)_{p,\xi} \cap \eta = (\eta)_{p,\xi}$. For every $\rho \in (\eta)_{p,\xi}$ there is some $P \in \mathcal{M}_p \cap M$ with $\delta_P \leq \xi$ such that $\rho \in \overline{P \cap \omega_2}$. Since $\mathcal{M}_p \cap M$ is finite there is such P with $\eta \in \overline{P \cap \omega_2}$. Since $P \in M$ it follows that $\overline{P} \subseteq M$, so $\eta \in M$ and therefore $\eta < \beta$. Finally, since η and β are p, ξ-connected, there is a chain $\eta = \gamma_0 < \gamma_1 < \ldots < \gamma_n = \beta$ such that γ_i and γ_{i+1} are p, ξ-linked, for all i. Let Q witness that $\eta = \gamma_0$ and γ_1 are p, ξ-linked. Then $\delta_Q \leq \xi$ and $\eta = \sup(Q \cap M \cap \omega_2)$. Since $\gamma_1 \in \overline{Q \cap \omega_2} \setminus M$ it follows that $Q \notin M$. Therefore, Q is as required. $\square$

We are now ready to define the distance function.

Definition 3.10. *Let $p = \mathcal{M}_p \in \mathbb{M}^*$, $\alpha, \beta \in \omega_2$, and $\xi \in \omega_1$. If $\alpha <_{p,\xi} \beta$ we define the p, ξ-distance of α and β as*

$$d_{p,\xi}(\alpha, \beta) = \text{o.t.}([\alpha, \beta)_{p,\xi}).$$

Otherwise we leave $d_{p,\xi}(\alpha, \beta)$ undefined.

Remark 3.11. Notice that for every p and ξ the function $d_{p,\xi}$ is additive, i.e. if $\alpha <_{p,\xi} \beta <_{p,\xi} \gamma$ then

$$d_{p,\xi}(\alpha, \gamma) = d_{p,\xi}(\alpha, \beta) + d_{p,\xi}(\beta, \gamma).$$

Moreover, we have that $d_{p,\xi}(\alpha, \beta) < g(\xi)$, for every $\alpha <_{p,\xi} \beta$.

We can now define the notion of forcing which adds an ω_2 chain in $(\omega_1^{\omega_1}, <_{\text{fin}})$ below the function g.

Definition 3.12. *Let $\mathbb{M}_3^*$ be the forcing notion whose elements are pairs $p = (f_p, \mathcal{M}_p)$, where f_p is a partial function from $\omega_2 \times \omega_1$ to ω_1, $\mathrm{dom}(f_p)$ is of the form $A_p \times F_p$ where $0 \in A_p \in [\omega_2]^{<\omega}$, $F_p \in [\omega_1]^{<\omega}$, $\mathcal{M}_p \in \mathbb{M}^*$, and for every $\alpha, \beta \in A_p$ with $\alpha < \beta$, every $\xi \in F_p$ and $M \in \mathcal{M}_p$:*

1. *$f_p(\alpha, \xi) < g(\xi)$,*
2. *if $\alpha <_{p,\xi} \beta$ then $f_p(\alpha, \xi) + d_{p,\xi}(\alpha, \beta) \leq f_p(\beta, \xi)$,*

We let $q \leq p$ if $f_p \subseteq f_q$, $\mathcal{M}_p \subseteq \mathcal{M}_q$ and for every $\alpha, \beta \in A_p$ and $\xi \in F_q \setminus F_p$ if $\alpha < \beta$ then $f_q(\alpha, \xi) < f_q(\beta, \xi)$.

We first show that for any $\alpha < \omega_2$ and $\xi < \omega_1$ any condition $p \in \mathbb{M}_3^*$ can be extended to a condition q such that $\alpha \in A_q$ and $\xi \in F_q$.

Lemma 3.13. *Let $p \in \mathbb{M}_3^*$ and $\delta \in \omega_2 \setminus A_p$. Then there is a condition $q \leq p$ such that $\delta \in A_q$.*

Proof. We let $\mathcal{M}_q = \mathcal{M}_p$, $A_q = A_p \cup \{\delta\}$ and $F_q = F_p$. On $A_p \times F_p$ we let f_q be equal to f_p. We need to define $f_q(\delta, \xi)$, for $\xi \in F_p$. Consider one such ξ. If δ does not belong to $\overline{M \cap \omega_2}$, for any $M \in \mathcal{M}_q$ with $\delta_M \leq \xi$, we can define $f_q(\delta, \xi)$ arbitrarily. Otherwise, we need to ensure that if $\alpha \in A_p$ and $\alpha <_{p,\xi} \delta$ then

$$f_{p,\xi}(\alpha, \xi) + d_{p,\xi}(\alpha, \delta) \leq f_q(\delta, \xi).$$

Similarly, if $\beta \in A_p$ and $\delta <_{p,\xi} \beta$ we have to ensure that

$$f_q(\delta, \xi) + d_{p,\xi}(\delta, \beta) \leq f_p(\beta, \xi).$$

By the additivity of $d_{p,\xi}$ we know that if $\alpha <_{p,\xi} \delta <_{p,\xi} \beta$ then $d_{p,\xi}(\alpha, \beta) = d_{p,\xi}(\alpha, \delta) + d_{p,\xi}(\delta, \beta)$. Since p is a condition we know that if $\alpha, \beta \in A_p$ then $f_p(\beta, \xi) \geq f_p(\alpha, \xi) + d_{p,\xi}(\alpha, \beta)$. Let α^* be the largest element of $A_p \cap (\delta)_{p,\xi}$. We can then simply define $f_q(\delta, \xi)$ by

$$f_q(\delta, \xi) = f_p(\alpha^*, \xi) + d_{p,\xi}(\alpha^*, \delta).$$

It is straightforward to check that the q thus defined is a condition. $\square$

Lemma 3.14. *Let $p \in \mathbb{M}_3^*$ and $\xi \in \omega_1 \setminus F_p$. Then there is a condition $q \leq p$ such that $\xi \in F_q$.*

Proof. We let $\mathcal{M}_q = \mathcal{M}_p$, $A_q = A_p$ and $F_q = F_p \cup \{\xi\}$. Then we need to extend f_p to $A_q \times \{\xi\}$. Notice that we now have the following commitments. Suppose $\alpha, \beta \in A_p$ and $\alpha < \beta$, then we need to ensure that $f_q(\alpha, \xi) < f_q(\beta, \xi)$ in order for q to be an extension of p. If in addition $\alpha <_{p,\xi} \beta$ then we need to ensure that

$$f_q(\alpha, \xi) + d_{q,\xi}(\alpha, \beta) \leq f_q(\beta, \xi)$$

in order for q to satisfy (2) of Definition 3.12. We define $f_q(\beta, \xi)$ by induction on $\beta \in A_q$ as follows. We let $f_q(0, \xi) = 0$. For $\beta > 0$ we let $f_q(\beta, \xi)$ be the maximum of the following set:

$$\{f_q(\alpha, \xi) + 1 : \alpha \in (A_q \cap \beta) \setminus (\beta)_{q,\xi}\} \cup \{f_q(\alpha, \xi) + d_{q,\xi}(\alpha, \beta) : \alpha \in A_q \cap (\beta)_{q,\xi}\}.$$

It is easy to see that $f_q(\beta, \xi) < g(\xi)$, for all $\beta \in A_q$, and that q is a condition extending p. $\square$

In order to prove strong properness of $\mathbb{M}_3^*$ we need to restrict to a relative club subset of $\mathcal{E}^2$ of elementary submodels of $H(\aleph_2)$ which are the restriction to $H(\aleph_2)$ of an elementary submodel of $H({2^{\aleph_1}}^+)$.

Definition 3.15. *Let $\mathcal{D}^2$ be the set of all $M \in \mathcal{E}^2$ such that $M = M^* \cap H(\aleph_2)$, for some $M^* \prec H(2^{\aleph_1 +})$. We let $\mathcal{D}^2_0 = \mathcal{D}^2 \cap \mathcal{E}^2_0$ and $\mathcal{D}^2_1 = \mathcal{D}^2 \cap \mathcal{E}^2_1$.*

We split the proof that $\mathbb{M}^*_3$ is $\mathcal{D}^2$-strongly proper in two lemmas.

Lemma 3.16. *Let $p \in \mathbb{M}^*_3$ and $M \in \mathcal{M}_p \cap \mathcal{D}^2_0$. Then p is an $(M, \mathbb{M}^*_3)$-strongly generic condition.*

Proof. Given $r \leq p$ we need to find a condition $r|M \in M$ such that every $q \leq r|M$ which is in M is compatible with r. By Lemma 3.14 we may assume that $\sup(P) \in A_r$, for every $P \in \mathcal{M}_r$. The idea is to choose $r|M$ which has the same type as r over some suitably chosen parameters in M. Let $D = \{\delta_P : P \in \mathcal{M}_r\} \cap M$. Since $M \in \mathcal{D}^2_0$ there is $M^* \prec H(2^{\aleph_1 +})$ such that $M = M^* \cap H(\aleph_2)$. By elementary of M^*, we can find in M an $\in$-chain $\mathcal{M}_{r^*} \in \mathbb{M}^*$ extending $\mathcal{M}_r \cap M$, a finite set $A_{r^*} \subseteq \omega_2$ and an order preserving bijection $\pi : A_r \to A_{r^*}$ such:

(1) π is the identity function on $A_r \cap M$,
(2) if $\alpha, \beta \in A_r$ then, for every $\xi \in D$,

$$d_{r^*,\xi}(\pi(\alpha), \pi(\beta)) = d_{r,\xi}(\alpha, \beta).$$

By Lemma 3.13 we can extend $f_r \restriction (A_r \cap M) \times (F_r \cap M)$ to a function $f_{r^*} : A_{r^*} \times (F_r \cap M) \to \omega_1$ such that $(f_{r^*}, \mathcal{M}_{r^*})$ is a condition in $\mathbb{M}^*_3$. Finally, we set $r|M = r^*$.

Suppose now $q \leq r|M$ and $q \in M$. We need to find a common extension s of q and r. We define $\mathcal{M}_s$ to be the closure under intersection of $\mathcal{M}_r \cup \mathcal{M}_q$. Indeed Lemma 2.17 shows that $\mathcal{M}_s \in \mathbb{M}^*$. We first compute the distance function $d_{s,\xi}$ in terms of $d_{r,\xi}$ and $d_{q,\xi}$, for $\xi < \omega_1$. First notice that the new models which are obtained by closing $\mathcal{M}_q \cup \mathcal{M}_r$ under intersection do not create new links and therefore do not influence the computation of the distance function.

Now, consider an ordinal $\xi < \omega_1$. If $\xi \geq \delta_M$ then all ordinals in $\overline{M \cap \omega_2}$ are pairwise r, ξ-linked. The countable models of $\mathcal{M}_q \setminus \mathcal{M}_r$ are all included in M so they do not add any new s, ξ-links. It follows that in this case $d_{s,\xi} = d_{r,\xi}$. Consider now an ordinal $\xi < \delta_M$. By Fact 3.8 if $\beta \in M$ then $(\beta)_{s,\xi} = (\beta)_{q,\xi}$. If $\beta \notin M$ then, by Fact 3.9 there is a $\eta \in A_r \cap M$ such that $(\beta)_{s,\xi} \cap M = (\eta)_{q,\xi}$. Let $\xi^* = \max(D \cap (\xi + 1))$. Then, again by Fact 3.9, η and β are r, ξ^*-connected and

$$d_{s,\xi}(\alpha, \beta) = d_{q,\xi}(\alpha, \eta) + d_{r,\xi^*}(\eta, \beta).$$

Let $A_s = A_q \cup A_r$ and $F_s = F_q \cup F_r$. Our next goal is to define an extension, call it f_s, of $f_q \cup f_r$ on $A_s \times F_s$. It remains to define f_s on

$$((A_q \setminus A_r) \times (F_r \setminus F_q)) \cup ((A_r \setminus A_q) \times (F_q \setminus F_r)).$$

Case 1: Consider first $\xi \in F_r \setminus F_q$ and let us define f_s on $(A_q \setminus A_r) \times \{\xi\}$. We already know that $d_{s,\xi} = d_{r,\xi}$, so we need to ensure that if $\alpha, \beta \in A_s$ and $\alpha <_{s,\xi} \beta$ then

$$f_s(\alpha, \xi) + d_{r,\xi}(\alpha, \beta) \leq f_s(\beta, \xi).$$

Notice that all the ordinals of A_q are r, ξ-linked as witnessed by M so then we will also have that for every $\alpha, \beta \in A_q$, if $\alpha < \beta$ then $f_s(\alpha, \xi) < f_s(\beta, \xi)$. In order to define $f_s(\alpha, \xi)$, for $\alpha \in A_q$, let α^* be the maximal element of $(\alpha)_{r,\xi} \cap A_r$ and let $f_s(\alpha, \xi) = f_r(\alpha^*, \xi) + d_{r,\xi}(\alpha^*, \alpha)$. It is straightforward to check that (2) of Definition 3.12 is satisfied in this case.

Case 2: Consider now some $\xi \in F_q \setminus F_r$. What we have to arrange is that $f_s(\alpha, \xi) < f_s(\beta, \xi)$, for every $\alpha, \beta \in A_r$ with $\alpha < \beta$. Moreover, for every $\alpha, \beta \in A_s$ with $\alpha <_{s,\xi} \beta$ we have to arrange that

$$f_s(\alpha, \xi) + d_{s,\xi}(\alpha, \beta) \leq f_s(\beta, \xi).$$

We define f_s on $(A_r \setminus A_q) \times \{\xi\}$ by setting

$$f_s(\beta, \xi) = f_q(\pi(\beta), \xi).$$

First, we show that the function $\alpha \mapsto f_s(\alpha, \xi)$ is order preserving on A_r. To see this observe that, since $q \leq r^* = r|M$ and $\xi \notin F_{r^*}$, the function $\alpha \mapsto f_q(\alpha, \xi)$ is strictly order preserving on A_{r^*}. Moreover, π is order preserving and the identity on $A_r \cap M = A_r \cap A_q$.

Assume now $\alpha, \beta \in A_s$ and $\alpha <_{s,\xi} \beta$. If $\alpha, \beta \in A_q$ then, since q is a condition, $f_s(\beta, \xi) \geq f_s(\alpha, \xi) + d_{q,\xi}(\alpha, \beta)$. On the other hand, we know that $d_{s,\xi}(\alpha, \beta) = d_{q,\xi}(\alpha, \beta)$, so we have the required inequality in this case. By Fact 3.8 the case $\alpha \in A_r \setminus A_q$ and $\beta \in A_q$ cannot happen. Suppose $\alpha \in A_q$ and $\beta \in A_r \setminus A_q$. Let $\xi^* = \max(D \cap (\xi + 1))$. By Fact 3.9 there is $\eta \in A_r \cap M$ such that

$$d_{s,\xi}(\alpha, \beta) = d_{s,\xi}(\alpha, \eta) + d_{r,\xi^*}(\eta, \beta).$$

By property (2) of π we have that $d_{r^*,\xi^*}(\eta, \pi(\beta)) = d_{r,\xi^*}(\eta, \beta)$. Since q extends r^* it follows that $d_{q,\xi^*}(\eta, \pi(\beta)) \geq d_{r^*,\xi^*}(\eta, \pi(\beta))$. Moreover, q is a condition and so:

$$f_q(\pi(\beta), \xi) \geq f_q(\alpha, \xi) + d_{q,\xi}(\alpha, \pi(\beta)) \geq f_q(\alpha, \xi) + d_{q,\xi^*}(\alpha, \pi(\beta)).$$

Therefore,

$$f_q(\pi(\beta), \xi) \geq f_q(\alpha, \xi) + d_{s,\xi}(\alpha, \beta).$$

The final case is when $\alpha, \beta \in A_r \setminus A_q$ and $\alpha <_{s,\xi} \beta$. Note that in this case, α and β are already r, ξ-connected, in fact, they are r, ξ^*-connected, where as before $\xi^* = \max(D \cap (\xi + 1))$. By property (2) of π we have that $\pi(\alpha)$ and $\pi(\beta)$ are r^*, ξ^*-connected and

$$d_{r^*,\xi^*}(\pi(\alpha), \pi(\beta) = d_{r,\xi}(\alpha, \beta).$$

Since $\xi^* \leq \xi$ and q extends r^* we have that

$$d_{q,\xi}(\pi(\alpha), \pi(\beta)) \geq d_{r^*,\xi^*}(\pi(\alpha), \pi(\beta)).$$

Since q is a condition we have

$$f_q(\pi(\beta), \xi) \geq f_q(\pi(\alpha), \xi) + d_{q,\xi}(\pi(\alpha), \pi(\beta)).$$

Since $d_{s,\xi}(\pi(\alpha), \pi(\beta)) = d_{q,\xi}(\pi(\alpha), \pi(\beta))$ we have $f_s(\beta, \xi) \geq f_s(\alpha, \xi) + d_{s,\xi}(\alpha, \beta)$, as required.

It follows that s is a condition which extends q and r. This completes the proof of Lemma 3.16. □

Lemma 3.17. *Let $p \in \mathbb{M}_3^*$ and $M \in \pi_1(\mathcal{M}_p)$. Then p is $(M, \mathbb{M}_3^*)$-strongly generic.*

Proof. Let $r \leq p$. We need to find a condition $r|M \in M$ such that any $q \leq r|M$ in M is compatible with r. We simply set

$$r|M = (f_r \restriction (A_r \times F_r) \cap M, \mathcal{M}_p \cap M).$$

We need to show that if $q \leq r|M$ is in M, then there is a condition $s \leq q, r$. Thanks to Lemma 2.17 we just need to define f_s, since we already know that $\mathcal{M}_r \cup \mathcal{M}_q$ is an $\in$-chain and belongs to $\mathbb{M}^*$. Since $\omega_1 \subseteq M$ we have that $F_r \subseteq M$ so we only need to define an extension f_s on $A_r \setminus A_q \times F_q \setminus F_r$. We know that $M \cap \omega_2$ is an initial segment of ω_2 so all the elements of $A_r \setminus A_q = A_r \setminus M$ are above all the ordinals of A_q. Given an ordinal $\xi \in F_q \setminus F_r$ we define $f_s(\beta, \xi)$, for $\beta \in A_r \setminus A_q$ by induction. We set:

$$f_s(\beta, \xi) = \max(\{f_s(\alpha, \xi)+1 : \alpha \in A_r \cap \beta\} \cup \{f_s(\alpha, \xi)+d_{s,\xi}(\alpha, \beta) : \alpha <_{s,\xi} \beta\}.$$

It is easy to check that $(f_s, \mathcal{M}_s)$ is a condition which extends both q and r. □

Corollary 3.18. *The forcing $\mathbb{M}_3^*$ is $\mathcal{D}^2$-strongly proper. Hence it preserves ω_1 and ω_2.* □

We have shown that for every $\alpha < \omega_2$ and $\xi < \omega_1$ the set

$$D_{\alpha,\xi} = \{p \in \mathbb{M}_3^* : \alpha \in A_p, \xi \in F_p\}$$

is dense in $\mathbb{M}_3$. If G is a V-generic filter in $\mathbb{M}_3^*$ we let

$$f_G = \bigcup\{f_p : p \in G\}.$$

It follows that $f_G : \omega_2 \times \omega_1 \to \omega_1$. For $\alpha < \omega_2$ we define $f_\alpha : \omega_1 \to \omega_1$ by letting $f_\alpha(\xi) = f_G(\alpha, \xi)$, for all ξ. It follows that the sequence $(f_\alpha : \alpha < \omega_2)$ is an increasing ω_2-chain in $(\omega_1^{\omega_1}, <_{\text{fin}})$. We have thus completed the proof of the following.

Theorem 3.19. *There is a $\mathcal{D}^2$-strongly proper forcing which adds an ω_2 chain in $(\omega_1^{\omega_1}, <_{\text{fin}})$.* □

4. Thin very tall superatomic Boolean algebras

A Boolean algebra $\mathcal{B}$ is called *superatomic* (sBa) iff every homomorphic image of $\mathcal{B}$ is atomic. In particular, $\mathcal{B}$ is an sBa iff its Stone space $S(\mathcal{B})$ is scattered. A very useful tool for studying scattered spaces is the Cantor-Bendixson derivative $A^{(\alpha)}$ of a set $A \subseteq S(\mathcal{B})$, defined by induction on α as follows. Let $A^{(0)} = A$, $A^{(\alpha+1)}$ is the set of limit points of $A^{(\alpha)}$, and $A^{(\lambda)} = \bigcap\{A^{(\alpha)} : \alpha < \lambda\}$, if λ is a limit ordinal. Then $S(\mathcal{B})$ is scattered iff for $S(\mathcal{B})^{(\alpha)} = \emptyset$, for some α.

When this notion is transferred to the Boolean algebra $\mathcal{B}$, we arrive at a sequence of ideals I_α, which we refer to as the Cantor-Bendixson ideals, defined by induction on α as follows. Let $I_0 = \{0\}$. Given I_α let $I_{\alpha+1}$ be generated by I_α together with all $b \in \mathcal{B}$ such that b/I_α, is an atom in $\mathcal{B}/I_\alpha$. If α is a limit ordinal, let $I_\alpha = \bigcup\{I_\xi : \xi < \alpha\}$. Then $\mathcal{B}$ is an sBa iff some $I_\alpha = \mathcal{B}$, for some α.

The height of an sBa $\mathcal{B}$, $\mathrm{ht}(\mathcal{B})$, is the least ordinal α such that $I_\alpha = \mathcal{B}$. For $\alpha < \mathrm{ht}\mathcal{B}$ let $\mathrm{wd}_\alpha(\mathcal{B})$ be the cardinality of the set of atoms in $\mathcal{B}/I_\alpha$. The *cardinal sequence* of $\mathcal{B}$ is the sequence $(\mathrm{wd}_\alpha(\mathcal{B}) : \alpha < \mathrm{ht}(\mathcal{B}))$. We say that $\mathcal{B}$ is *κ-thin-very tall* if $\mathrm{ht}(\mathcal{B}) = \kappa^{++}$ and $\mathrm{wd}_\alpha(\mathcal{B}) = \kappa$, for all $\alpha < \kappa^{++}$. If $\kappa = \omega$ we simply say that $\mathcal{B}$ is *thin very tall*.

Baumgartner and Shelah [2] constructed a forcing notion which adds a thin very tall sBa. This is achieved in two steps. First they adjoin by a σ-closed $\aleph_2$-cc forcing a function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ with some special properties. Such a function is called a Δ-function. In the second step they use a Δ-function to define a ccc forcing notion which adds a thin very tall sBa. The purpose of this section is to show how this can be achieved directly by using generalizes side conditions. The following concept from [2] was made explicit by Bagaria in [1].

Definition 4.1. *Given a cardinal sequence $\theta = \langle\kappa_\alpha : \alpha < \lambda\rangle$, where each κ_α is an infinite cardinal, we say that a structure $(T, \leq, i)$ is a θ-*poset *if $<$ is a partial ordering on T and the following hold:*

1. *$T = \bigcup\{T_\alpha : \alpha < \lambda\}$, where each T_α is of the form $\{\alpha\} \times Y_\alpha$, and Y_α is a set of cardinality κ_α.*
2. *If $s \in T_\alpha$, $t \in T_\beta$ and $s < t$, then $\alpha < \beta$.*
3. *For every $\alpha < \beta < \lambda$, if $t \in T_\beta$ then the set $\{s \in T_\alpha : s < t\}$ is infinite.*
4. *i is a function from $[T]^2$ to $[T]^{<\omega}$ with the following properties:*
    - (a) *If $u \in i\{s, t\}$, then $u \leq s, t$*
    - (b) *If $u \leq s, t$, then there exists $v \in i\{s, t\}$ such that $u \leq v$.*

We let $\Omega(\lambda)$ denote the sequence of length λ with all entries equal to ω. The following is implicitly due to Baumgartner (see [1] for a proof).

Fact 4.2. *Let $\theta = \langle \kappa_\alpha : \alpha < \lambda \rangle$ be a sequence of cardinals. If there exists a θ-poset, then there exists an sBa whose cardinal sequence is θ.* $\square$

We now define a forcing notion which adds an $\Omega(\omega_2)$-poset. If $x \in \omega_2 \times \omega$ is of the form (α, n) then we denote α by α_x and n by n_x.

Definition 4.3. *Let $\mathbb{M}_4$ be the forcing notion whose elements are tuples $p = (x_p, \leq_p, i_p, \mathcal{M}_p)$, where x_p is a finite subset of $\omega_2 \times \omega$, $\leq_p$ is a partial ordering on x_p, $i_p : [x_p]^2 \to [x_p]^{<\omega}$, $\mathcal{M}_p \in \mathbb{M}$ and the following hold:*

1. *if $s, t \in x_p$ and $s <_p t$ then $\alpha_s < \alpha_t$,*
2. *if $u \in i_p\{s, t\}$ then $u \leq_p s, t$,*
3. *for every $u \leq_p s, t$ there is $v \in i_p\{s, t\}$ such that $u \leq_p v$,*
4. *for every $s, t \in x_p$ and $M \in \mathcal{M}_p$ if $s, t \in M$ then $i_p\{s, t\} \in M$.*

We let $q \leq p$ if and only if $x_q \supseteq x_p$, $\leq_q \restriction x_p = \leq_p$, $i_q \restriction [x_p]^2 = i_p$ and $\mathcal{M}_p \subseteq \mathcal{M}_q$.

We first observe that a version of Lemma 1.8 holds for $\mathbb{M}_4$.

Lemma 4.4. *Let $M \in \mathcal{E}^2$ and let $p \in \mathbb{M}_4 \cap M$. Then there is a new condition, which we will call p^M, that is the smallest element of $\mathbb{M}_4$ extending p such that $M \in \mathcal{M}_{p^M}$.*

Proof. If $M \in \mathcal{E}^2_1$ then simply let $p^M = (x_p, \leq_p, i_p, \mathcal{M}_p \cup \{M\})$. If $M \in \mathcal{E}^2_0$, then, as in Lemma 1.8, we let $\mathcal{M}_{p^M}$ be the closure of $\mathcal{M}_p \cup \{M\}$ under intersection and let $p^M = (x_p, \leq_p, i_p, \mathcal{M}_{p^M})$. We need to check that condition (4) of Definition 4.3 is satisfied. Since $p \in M$ we have that $x_p \subseteq M$. In the case $M \in \mathcal{E}^2_1$ the only new model in $\mathcal{M}_{p^M}$ is M so condition (4) holds for p^M since it holds for p. In the case $M \in \mathcal{E}^2_0$ there are also models of the form $N \cap M$, where $N \in \pi_1(\mathcal{M}_p)$. However, condition (4) holds for both N and M and so it holds for their intersection. $\square$

Next, we show that $\mathbb{M}_4$ is $\mathcal{E}^2$-proper. We split this in two parts.

Lemma 4.5. $\mathbb{M}_4$ *is $\mathcal{E}^2_0$-proper.*

Proof. Let θ be a sufficiently large regular cardinal and let M^* be a countable elementary submodel of $H(\theta)$ containing all the relevant objects. Then $M = M^* \cap H(\omega_2)$ belongs to $\mathcal{E}^2_0$. Suppose $p \in \mathbb{M}_4 \cap M$. Let p^M be the condition defined in Lemma 4.4, i.e. $p^M = (x_p, \leq_p, i_p, \mathcal{M}_{p^M})$, where $\mathcal{M}_{p^M}$ is the closure of $\mathcal{M}_p \cup \{M\}$ under intersection. We show that p^M is $(M^*, \mathbb{M}_4)$-generic. Let $D \in M^*$ be a dense subset of $\mathbb{M}_4$ and $r \leq p^M$. We need to find a condition $q \in D \cap M^*$ which is compatible with r. Note that we may assume that $r \in D$. We define a condition $r|M$ as follows. First let $x_{r|M} = x_r \cap M$ and then let $\leq_{r|M} = \leq_r \restriction x_{r|M}$ and $i_{r|M} = i_r \restriction [x_{r|M}]^2$. Condition (4) of Definition 4.3 guarantees that if $s, t \in x_{r|M}$ then $i_r\{s, t\} \subseteq M$.

Finally, let $\mathcal{M}_{r|M} = \mathcal{M}_r \cap M$. It follows that $r|M = (x_{r|M}, i_{r|M}, i_{r|M}, \mathcal{M}_{r|M})$ belongs to $\mathbb{M}_4 \cap M$. By elementarity of M^* in $H(\theta)$ there is a condition $q \in D \cap M^*$ extending $r|M$ such that $(x_q \setminus x_{r|M}) \cap N = \emptyset$, for all $N \in \pi_0(\mathcal{M}_{r|M})$.

Claim 4.6. *q and r are compatible.*

Proof. We define a condition s as follows. We set $x_s = x_q \cup x_r$ and we let $\leq_s$ be the transitive closure of $\leq_q \cup \leq_r$, i.e. if $u \in x_q \setminus x_r$, $v \in x_r \setminus x_q$ and $t \in x_{r|M}$ are such that $u \leq_q t$ and $t \leq_r v$, then we let $u \leq_s v$. Similarly, if $v \leq_r t$ and $t \leq_q u$ we let $v \leq_s u$. We let $\mathcal{M}_s$ be the closure under intersection of $\mathcal{M}_q \cup \mathcal{M}_r$. It remains to define i_s. For $z \in x_r$ let $A_z = \{t \in x_{r|M} : t \leq_r z\}$ and for $z \in x_q$ let $B_z = \{t \in x_{r|M} : t \leq_q z\}$. We let

$$i_s\{u,v\} = \begin{cases} i_q\{u,v\} & \text{if } u,v \in x_q, \\ i_r\{u,v\} & \text{if } u,v \in x_r, \\ \bigcup_{t \in A_v} i_q\{u,t\} \cup \bigcup_{t \in B_u} i_r\{t,v\} & \text{if } u \in x_q \setminus x_r \text{ and } v \in x_r \setminus x_q. \end{cases}$$

We now need to check property (4) of Definition 4.3, i.e. for every $u, v \in x_s$ and $P \in \mathcal{M}_s$, if $u, v \in P$ then $i_s\{u,v\} \in P$. First of all notice that we only need to show the above property for $P \in \mathcal{M}_q \cup \mathcal{M}_r$, because the other models in $\mathcal{M}_s$ are obtained by intersection and, if (4) holds for u, v and P and also for u, v and Q, it also holds for u, v and $P \cap Q$.

Case 1: $u, v \in x_q$ and $P \in \mathcal{M}_r$. If $P \in \mathcal{M}_{r|M}$ then $P \in \mathcal{M}_q$ and then (4) holds since q is a condition and $i_s\{u,v\} = i_q\{u,v\}$. Suppose now $P \in \mathcal{M}_r \setminus M$ and $\delta_R < \delta_M$. Then, by Fact 3.3, $P \cap M \in \mathcal{M}_{r|M}$ and so $P \cap M \in \mathcal{M}_q$, therefore (4) of Definition 4.3 holds again. Finally, if $\delta_R \geq \delta_M$ then, by Fact 3.2, $P \cap M \cap \omega_2$ is an initial segment of $M \cap \omega_2$. We know that $i_q\{u,v\} \in M$ and for every $w \in i_q\{u,v\}$ $\alpha_w \leq \min(\alpha_u, \alpha_v)$. Therefore, we have that $i_q\{u,v\} \in P \cap M$. Since $i_s\{u,v\} = i_q\{u,v\}$, we conclude that (4) holds in this case.

Case 2: $u, v \in x_r$ and $P \in \mathcal{M}_q$. If $u, v \in x_{r|M}$ then $u, v \in x_q$ and again, since q is a condition and $i_s\{u,v\} = i_q\{u,v\}$, we know that $i_s\{u,v\} \in P$. Suppose now that u and v are not both in M. If $P \in \mathcal{E}^2_0$ then $P \subseteq M$ and so we cannot have $u, v \in P$. If $P \in \mathcal{E}^2_1$ we know that $P \cap \omega_2$ is an initial segment of ω_2. Moreover, if $w \in i_s\{u,v\}$ then $\alpha_w \leq \min(\alpha_u, \alpha_v)$ and so if $u, v \in P$ we also have that $i_s\{u,v\} \in P$, so (4) of Definition 4.3 holds again.

Case 3: $u \in x_q \setminus x_r$ and $v \in x_r \setminus x_q$. If $P \in \mathcal{E}^2_1$ then $P \cap \omega_2$ is an initial segment of ω_2. Moreover, as before, we have that $\alpha_w \leq \min(\alpha_u, \alpha_v)$, for every $w \in i_s\{u,v\}$. Therefore, $i_s\{u,v\} \in P$. Suppose now $P \in \mathcal{E}^2_0$. If

$P \in \pi_0(\mathcal{M}_q)$ then $P \subseteq M$ so $v \notin P$. Now assume $P \in \pi_0(\mathcal{M}_r)$. If $\delta_R < \delta_M$ then by Fact 3.3, $P \cap M \in \mathcal{M}_{r|M}$. However, the condition q is chosen so that $(x_q \setminus x_{r|M}) \cap N$, for all $N \in \pi_0(\mathcal{M}_{r|M})$, therefore in this case $u \notin P$. Assume now $\delta_R \geq \delta_M$. Then, by Fact 3.2, we have that $P \cap M \cap \omega_2$ is an initial segment of $M \cap \omega_2$. Consider first some $t \in A_v$. Then $t \in M$ and $i_q\{u,t\} \in M$. Moreover, $\alpha_w \leq \min(\alpha_u, \alpha_t)$, for every $w \in i_q\{u,t\}$. Since $P \cap M \cap \omega_2$ is an initial segment of $M \cap \omega_2$, it follows that $\alpha_w \in P \cap M$, for every $w \in i_q\{u,t\}$. This implies that $i_q\{u,t\} \in P \cap M$. Finally, consider some $t \in B_u$. Then $t \in M$ and, since $u \in P \cap M$, $\alpha_t \leq \alpha_u$ and $P \cap M \cap \omega_2$ is an initial segment of $M \cap \omega_2$, we have that $\alpha_t \in P \cap M$ and so $t \in P \cap M$. Now, since r is a condition, $P \in \mathcal{M}_r$ and $t, v \in P$, we have that $i_r\{t,v\} \in P$, so (4) of Definition 4.3 holds in this case as well.

It follows that s is a condition which extends both q and r. This completes the proof of Claim 4.6 and Lemma 4.5. □

□

Lemma 4.7. $\mathbb{M}_4$ *is* $\mathcal{E}_1^2$*-proper.*

Proof. Let θ be a sufficiently large regular cardinal and M^* an elementary submodel of $H(\theta)$ containing all the relevant objects such that $M = M^* \cap H(\omega_2)$ belongs to $\mathcal{E}_1^2$. Fix $p \in M \cap \mathbb{M}_4$. Let p^M be as in Lemma 4.4. We claim that p^M is $(M^*, \mathbb{M}_4)$-generic. In order to verify this consider a dense subset D of $\mathbb{M}_4$ which belongs to M^* and a condition $r \leq p^M$. We need to find a condition $q \in D \cap M^*$ which is compatible with r. By extending r if necessary we may assume it belongs to D. Let $r|M = (x_{r|M}, i_{r|M}, i_{r|M}, \mathcal{M}_{r|M})$ be as in Lemma 4.5. By elementarity of M^* in $H(\theta)$, we can find $q \leq r|M$, in $D \cap M$, such that $(x_q \setminus x_{r|M}) \cap N = \emptyset$, for all $N \in \mathcal{M}_{r|M}$, and if $u \in x_{r|M}$ and $v \in x_q \setminus x_{r|M}$ then $\alpha_u < \alpha_v$.

Claim 4.8. q *and* r *are compatible.*

Proof. We define a condition s as follows. We set $x_s = x_q \cup x_r$, $\leq_s = \leq_q \cup \leq_r$ and $\mathcal{M}_s = \mathcal{M}_q \cup \mathcal{M}_r$. Note that $\leq_s$ is a partial order and $\mathcal{M}_s \in \mathbb{M}$. It remains to define i_s. We let

$$i_s\{u,v\} = \begin{cases} i_q\{u,v\} & \text{if } u,v \in x_q, \\ i_r\{u,v\} & \text{if } u,v \in x_r, \\ \{z \in x_{r|M} : z \leq_q u \text{ and } z \leq_r v\} & \text{if } u \in x_q \setminus x_r,\ v \in x_r \setminus x_q. \end{cases}$$

We need to check (4) of Definition 4.3. So, suppose $u, v \in x_s$, $P \in \mathcal{M}_s$ and $u, v \in P$. We need to show that $i_s\{u,v\} \in P$.

Case 1: $u, v \in x_q$ and $P \in \mathcal{M}_r$. If $P \cap M \in \mathcal{M}_{r|M}$ this follows from the fact that q is a condition and $\mathcal{M}_{r|M} \subseteq \mathcal{M}_q$. If $P \cap M \notin \mathcal{M}_{r|M}$ then $M \subseteq P$ and, since $i_q\{u,v\} \in M$, it follows that $i_s\{u,v\} \in P$.

Case 2: $u, v \in x_r$ and $P \in \mathcal{M}_q$. If $u, v \in M$ then $u, v \in x_q$, so $i_s\{u, v\} \in P$ follows from the fact that q is a condition. Assume now, for concreteness, that $v \notin M$. If $P \in \mathcal{M}_q$ then $v \notin P$ and if $P \in \mathcal{M}_r$ then (4) of Definition 4.3 follows from the fact that r is a condition.

Case 3: $u \in x_q \setminus x_r$ and $v \in x_r \setminus x_q$. If $P \in \mathcal{M}_q$ then $v \notin P$. If $P \in \mathcal{M}_r$ then either $P \cap M \in \mathcal{M}_{r|M}$ and then, by the choice of q, we have that $u \notin P$. Otherwise $M \subseteq P$ and in this case $x_{r|M} \subseteq P$. Since $i_s\{u, v\} \subseteq x_{r|M}$ it follows that (4) of Definition 4.3 holds in this case as well. □

This completes the proof of Lemma 4.7. □

Corollary 4.9. *The forcing* $\mathbb{M}_4$ *is* $\mathcal{E}^2$*-proper. Hence it preserves* ω_1 *and* ω_2. □

It is easy to see that the set

$$D_{\alpha,n} = \{p \in \mathbb{M}_4 : (\alpha, n) \in x_p\}$$

is dense in $\mathbb{M}_4$, for every $\alpha \in \omega_2$ and $n \in \omega$. Moreover, given $t \in \omega_2 \times \omega$, $\eta < \alpha_t$ and $n < \omega$, one verifies easily that the set

$$E_{t,\eta,n} = \{p : t \in x_p \text{ and } |\{i : (\eta, i) \in x_p \text{ and } (\eta, i) \leq_p t\}| \geq n\}$$

is dense. Then if G is V-generic filter on $\mathbb{M}_4$ let

$$\leq_G = \bigcup \{\leq_p : p \in G\} \qquad \text{and} \qquad i_G = \bigcup \{i_p : p \in G\}.$$

It follows that $(\omega_2 \times \omega, \leq_G, i_G)$ is an $\Omega(\omega_2)$-poset in $V[G]$. We have therefore proved the following.

Theorem 4.10. *There is an* $\mathcal{E}^2$*-proper forcing notion which adds an* $\Omega(\omega_2)$*-poset.* □

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