21-624 Topics in analysis: classical descriptive set theory
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Descriptive set theory combines analysis and logic. There are two standard texts, both of which are recommended but not required. Kechris' book is written for those with beginning graduate level knowledge of general topology, measure theory, functional analysis, as well as some set theory. (The appendices provide a short review of the set theory needed.) Moschovakis' book is written for first or second year graduate students in logic, especially those interested in set theory or computability. Our approach will depend on the background of the students and will include topics from both books.

Descriptive set theory is the study of separable complete metric space, which are called Polish spaces. For example, $\mathbb{R}$ is a Polish space! The definable subsets of Polish spaces are classified according to the complexity of their definitions. Properties are proved by induction along the complexity hierarchy. The Borel subsets of $\mathbb{R}^n$ form an initial segment of the hierarchy. However, the notation commonly used by analysts, namely $\mathcal{G}$, $\mathcal{F}$, $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$, $\mathcal{G}_{\delta\sigma}$, $\mathcal{F}_{\sigma\delta}$, etc., is inadequate because the Borel hierarchy has uncountably many levels. Instead, we will use the notation $\mathbf{\Sigma}^{\mathbf{0}}_\alpha$ and $\mathbf{\Pi}^{\mathbf{0}}_\alpha$ for the pair of classes at the α'th level of the Borel hierarchy, where the index α varies over countable ordinals. Beyond the Borel hierarchy is the projective hierarchy, which is defined as follows. If there is a Borel set $B \subseteq \mathbb{R}^{n+1}$ such that

$$A = \{\mathbf{x} \in \mathbb{R}^n \mid \text{there exists } y \in \mathbb{R} \text{ such that } (\mathbf{x}, y) \in B\},$$

then A is called analytic or $\mathbf{\Sigma}^{\mathbf{1}}_{\mathbf{1}}$. In general, if $k, n \in \mathbb{N}$ and $A \subseteq \mathbb{R}^n$, then

- A is $\mathbf{\Pi}^{\mathbf{1}}_{\mathbf{k}} \iff \mathbb{R}^n - A$ is $\mathbf{\Sigma}^{\mathbf{1}}_{\mathbf{k}}$
- A is $\mathbf{\Sigma}^{\mathbf{1}}_{\mathbf{k+1}} \iff$ there is a $\mathbf{\Pi}^{\mathbf{1}}_{\mathbf{k}}$ set $B \subseteq \mathbb{R}^{n+1}$ such that

$$A = \{\mathbf{x} \in \mathbb{R}^n \mid \text{there exists } y \in \mathbb{R} \text{ such that } (\mathbf{x}, y) \in B\}.$$

Consider the following questions. Is every analytic (i.e., $\mathbf{\Sigma}^{\mathbf{1}}_{\mathbf{1}}$) subset of $\mathbb{R}$ Lebesgue measurable? Yes, this is a classical theorem. Is every projection of a co-analytic (i.e., $\mathbf{\Sigma}^{\mathbf{1}}_{\mathbf{2}}$) subset of $\mathbb{R}$ Lebesgue measurable? The answer is neither "yes" nor "no" and requires logic to understand.

Recommended texts:

- Alexander S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag Graduate Texts in Mathematics 156
- Yiannis N. Moschovakis, *Descriptive Set Theory*, North-Holland Studies in Logic 100