MA 355 Homework 6 solutions

#1 Prove the sequence $s_1 = 1,\ s_{n+1} = \frac{1}{4}(s_n + 5)$ where $n \in \mathbb{N}$ is monotone and bounded. Then find the limit.

Claim: Bounded above by 2 and below by 0. The lower bound is clear. Prove the upper bound by induction: We know $s_1 = 1$. Assume $s_n \le 2$ for some n. Then $s_{n+1} = \frac{1}{4}(s_n+5) \le \frac{1}{4}(2+5) < 2$. Also prove monotone by induction: $s_2 - s_1 = \frac{1}{4} * 6 - 1 = \frac{1}{2} > 0$. Thus $s_2 > s_1$. Now assume $s_n - s_{n-1} > 0$. Then $s_{n+1} - s_n = \frac{1}{4}(s_n - s_{n-1}) > 0$. Therefore $s_{n+1} > s_n$ and the sequence is monotone. Since monotone and bounded we know the limit exists. Using the fact that $\lim s_n = \lim s_{n-1} = L$ we find $L = \frac{5}{3}$.

#2 Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0, \quad s_{2m} = \frac{s_{2m-1}}{2}, \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Pf: We can show by induction that

$$s_{2m} = \frac{2^{m-1} - 1}{2^m}, \quad s_{2m+1} = \frac{2^m - 1}{2^m}, \quad m \in \mathbb{N}$$

It follows the subsequences $\{s_{2m}\}$ consisting of even terms tends to $\frac{1}{2}$ and the subsequence $\{s_{2m+1}\}$ consisting of odd terms tends to 1. Therefore the sequence $\{s_n\}$ has exactly two limit points $\frac{1}{2}$ (the lower limit) and 1 (the upper limit).

#3 Find an example of a sequence of real numbers satisfying each set of properties:

a) Cauchy but not monotone, $\{\frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, ...\}$

b) Monotone but not Cauchy, $a_n = n$

c) Bounded but not Cauchy, $x_n = (-1)^n$

#4 For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\limsup_{n\to\infty} (a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided the sum on the right is not of the form $\infty - \infty$. Pf: If either lim sup on the right hand side is $+\infty$, then the inequality is trivially satisfied. Also, if $\limsup_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ tends to $-\infty$; if $\limsup_{n\to\infty} b_n < \infty$ then the sequence $a_n + b_n$ tends to $-\infty$ as well. So, it remains to consider the case when both $\limsup a_n$ and $\limsup b_n$ are finite.

Take any subsequence $\{a_{n_k} + b_{n_k}\}$ that tends to $\limsup_{n\to\infty}\{a_n + b_n\}$. Since $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, we conclude that $\limsup_{k\to\infty} a_{n_k} \le \limsup_{n\to\infty} a_n$. Let $\{a_{n_{k_\ell}}\}$ be the subsequence of a_{n_k} that tends to $\limsup_{k\to\infty} a_{n_k}$. Similarly, since $\limsup_{\ell\to\infty} b_{n_{k_\ell}} \le \limsup_{n\to\infty} b_n$, we obtain

$$\limsup(a_n + b_n) = \lim(a_{n_k} + b_{n_k}) = \lim_{\ell\to\infty}(a_{n_{k_\ell}} + b_{n_{k_\ell}}) = \limsup a_{n_k} + \limsup b_{n_k} \le \limsup a_n + \limsup b_n.$$

#5 Prove: A monotone decreasing sequence is convergent iff it is bounded.

Pf: Suppose $\{s_n\}$ is a bounded decreasing sequence. Let S denote the nonempty bounded set $\{s_n : n \in \mathbb{N}\}$. By the completeness axiom, S has a greatest lower bound, let $s = \inf S$. We claim $\lim s_n = s$. Given any $\varepsilon > 0$, $s + \varepsilon$ is not an upper bound for S. Thus $\exists N \in \mathbb{N}$ such that $s_N < s + \varepsilon$. Furthermore, since $\{s_n\}$ is decreasing and s is a lower bound for S, we have $s \le s_n \le s_N < s + \varepsilon$. for all $n > N$. Hence $s_n \to s$.