A SHARP COMMUTATOR ESTIMATE FOR SUB-COULOMB RIESZ MODULATED ENERGIES

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Abstract. We prove a functional inequality in any dimension controlling the derivative along a transport of the sub-Coulomb Riesz modulated energy in terms of the modulated energy itself. This modulated energy was introduced by the third author and collaborators in the study of mean-field limits and statistical mechanics of Coulomb/Riesz gases, where this control is an essential ingredient. Previous work of the last two authors and Q.H. Nguyen [NRS22] showed a similar functional inequality but with an additive N-dependent error which is not sharp. In this paper, we obtain the optimal $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$ additive error. Our method is conceptually simple and (like previous work) relies on the observation that the derivative along a transport of the modulated energy is the quadratic form of a commutator. Through a new potential truncation scheme based on a wavelet-type representation of the Riesz potential to handle its singularity, the proof reduces to "averaging" over a family of Kato-Ponce type estimates.

The commutator estimate has applications to sharp rates of convergence for mean-field limits, quasi-neutral limits, and central limit theorems for the fluctuations of Coulomb/Riesz gases both at and out of thermal equilibrium. In particular, we show here for $\mathsf{s} < \mathsf{d} - 2$ the expected $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$-rate in the modulated energy distance for the mean-field convergence of first-order Hamiltonian and gradient flows. This complements the recent work [RS23b] on the optimal rate for the (super-)Coulomb case $\mathsf{d} - 2 \le \mathsf{s} < \mathsf{d}$ and therefore resolves the entire Riesz case.

1. Introduction

1.1. **Motivation.** In any dimension d, consider the class of *Riesz* interactions

$$\mathsf{g}(x) = \begin{cases} \frac{1}{\mathsf{s}}|x|^{-\mathsf{s}}, & \mathsf{s} \neq 0 \\ -\log|x|, & \mathsf{s} = 0, \end{cases} \tag{1.1}$$

with the assumption that $0 \le \mathsf{s} < \mathsf{d} - 2$. Up to a normalizing constant $\mathsf{c}_{\mathsf{d},\mathsf{s}}$, these interactions are characterized as fundamental solutions of the fractional Laplacian: $(-\Delta)^{\frac{\mathsf{d}-\mathsf{s}}{2}}\mathsf{g} = \mathsf{c}_{\mathsf{d},\mathsf{s}}\delta_0$. The particular case $\mathsf{s} = \mathsf{d} - 2$ corresponds to the classical *Coulomb* interaction from physics, and thus $\mathsf{s} < \mathsf{d} - 2$ means that we are considering the *sub-Coulomb* case. The restriction to $\mathsf{s} \ge 0$ is to only consider the more challenging singular case, while the restriction $\mathsf{s} < \mathsf{d}$ is to exclude the *hypersingular* case, which is not of the mean-field type considered in this paper (e.g. see [HSST20]).

Given a system of N distinct points $X_N = (x_1, \ldots, x_N) \in (\mathbb{R}^{\mathsf{d}})^N$ with interaction energy

$$\sum_{1 \le i \neq j \le N} \mathsf{g}(x_i - x_j), \tag{1.2}$$

one is led to comparing the sequence of empirical measures $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ to an average, or *mean-field*, density μ. This comparison is conveniently performed by considering a *modulated*

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energy, or Coulomb/Riesz (squared) "distance" between μ_N and μ, defined by

$$\mathsf{F}_N(X_N,\mu) := \frac{1}{2}\int_{(\mathbb{R}^{\mathsf{d}})^2\setminus\triangle} \mathsf{g}(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\Big)^{\otimes 2}(x,y). \tag{1.3}$$

We remove the diagonal $\triangle$ from the domain of integration in order to remove the infinite self-interaction of each particle. This object originated in the study of the statistical mechanics of Coulomb and Riesz gases [SS15a, SS15b, RS16, PS17] and later was used in the derivation of mean-field dynamics [Due16, Ser20, NRS22] and following works. We refer to [Ser24, Chapter 4] for a comprehensive discussion of the modulated energy.

An essential point is to control quantities that correspond to differentiating F_N along a transport $v:\mathbb{R}^{\mathsf{d}}\to\mathbb{R}^{\mathsf{d}}$, i.e. the quantities

$$\frac{d^n}{dt^n}\Big|_{t=0}\mathsf{F}_N((\mathbb{I}+tv)^{\oplus N}(X_N),(\mathbb{I}+tv)\#\mu) = \frac{1}{2}\int_{(\mathbb{R}^{\mathsf{d}})^2\setminus\triangle}\nabla^{\otimes n}\mathsf{g}(x-y):(v(x)-v(y))^{\otimes n}d(\mu_N-\mu)^{\otimes 2}(x,y), \tag{1.4}$$

where $\mathbb{I}:\mathbb{R}^{\mathsf{d}}\to\mathbb{R}^{\mathsf{d}}$ is the identity, $(\mathbb{I}+tv)^{\oplus N}(X_N) := (x_1+tv(x_1),\ldots,x_N+tv(x_N))$, and $:$ denotes the inner product between the tensors.For applications to the mean-field limit, V is the velocity field of the limiting evolution as $N\to\infty$. For applications to central limit theorems (CLTs) for the fluctuations (the next-order description), v is the gradient of a test function evolved along the adjoint linearized mean-field flow [HRS, **?**], and these inequalities are that core of the "transport" approach to fluctuations of canonical Gibbs ensembles [LS18, BLS18, Ser23, **?**]. We also mention the quantity in (1.4) is also the same that appears in the loop or Dyson-Schwinger equations in the random matrix theory literature, e.g. [BG13, BG24, BBNY19], which are another avatar of the transport method.

A key question, which we address in this paper, is to establish a functional inequality asserting that the quantity in (1.4) is always bounded by $C_1(\mathsf{F}_N(X_N,\mu)+C_2N^{-\alpha})$ with $\alpha>0$, where $C_1>0$ is a constant depending on d,s,v and $C_2>0$ depends on of $\mathsf{d},\mathsf{s},\mu$. For $n=1$, this was first proved in [LS18] in the 2D Coulomb case, then generalized to the super-Coulomb case, with $\max\{\mathsf{d}-2,0\}\le\mathsf{s}<\mathsf{d}$, in [Ser20]. The proof relied on identifying a *stress-energy tensor* in (1.4) and using integration by parts. Subsequently, the second author [Ros20] observed that the expressions (1.4) may also be viewed as the quadratic form of a *commutator*, akin to the famous Calderón commutator [Cal80]. This perspective was then used by the last two authors and Q.H. Nguyen [NRS22] to show the desired functional inequalities for all values of s and any order[1] $n\ge 2$ and applying to a broader class of g's that may be regarded as perturbations of Riesz interactions, including Lennard-Jones type potentials. Although these stress-energy and commutator perspectives may seem distinct, they are in fact dual to each other [RS24c]. Not only are these functional inequalities have been crucial for proving CLTs for the fluctuations of Riesz gases [LS18, Ser23, **?**], and even more so for deriving mean-field limits [Ser20, Ros22b, Ros22a, NRS22, dCRS23b, RS24c] and large deviation principles [HC23]. The inequalities have been further used to show joint classical and mean-field limits [GP22] and supercritical mean-field limits of classical [HKI21, Ros23, M2́4, RS24a] and quantum systems of particles [Ros21, Por23].

An essential question for these inequalities is the optimal size of the additive error, i.e. the exponent α which a priori depends on d,s. The appropriate comparison is with the minimal size of $\mathsf{F}_N(X_N,\mu)$. In the (super-)Coulomb case, it is known that $\mathsf{F}(X_N,\mu)+\frac{\log(N\|\mu\|_{L^\infty})}{2\mathsf{d}N}\mathbf{1}_{\mathsf{s}=0}\ge -C\|\mu\|_{L^\infty}N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$ and this lower bound is sharp.[2] For instance, see [**?**, Sections 4.2, 12.4]. We show

[1] Strictly speaking, the paper only explicitly considers the case $n=2$, but the argument extends (with more algebra) to any $n\ge 2$.

[2] The term $\propto\frac{\log N}{N}$ in the $\mathsf{s}=0$ case should be viewed as an error, but rather the fact that the right quantity to consider is $\mathsf{F}(X_N,\mu)+\frac{\log(N\|\mu\|_{L^\infty})}{2\mathsf{d}N}\mathbf{1}_{\mathsf{s}=0}$.

in this work that the same lower bound holds in the sub-Coulomb case as well (see Remark 2.12), which was not previously known. Moreover, this lower bound is in general sharp [HSSS17]. In controlling the magnitude of (1.4), one needs a right-hand side which is also nonnegative. Thus, the best error one may hope for is of size $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$. Here, optimality is understood in the "worst-case" sense, i.e. there exists a point configuration X_N such that achieves the bound. This should be contrasted with the "average" sense, where $X_N \sim f_N \in \mathcal{P}((\mathbb{R}^{\mathsf{d}})^N))$ and one estimates $\mathbb{E}_{X_N\sim f_N}[|(1.4)|]$. From the perspective of statistical mechanics, the worst-case corresponds to zero-temperature, while the average case encompasses a range of temperatures, with the infinite temperature limit corresponding to iid points. It is well-known that averaging may produce smaller errors. One quickly checks that if $f_N \sim \mu^{\otimes N}$ (i.e. μ-iid), then $\mathbb{E}_{X_N\sim f_N}[|(1.4)|] \approx \frac{1}{N}$. Moreover, it is known that for bounded interaction potentials, there are high-temperature/entropic commutator estimates in which $\mathbb{E}_{X_N\sim f_N}[|(1.4)|]$ is controlled by a right-hand side consisting of the normalized relative entropy $\frac{1}{N}H(f_N|\mu^{\otimes N}) + O(\frac{1}{N})$ [JW18, LLN20]; and in fact, versions of such estimates with a $\frac{1}{N}$ additive error also hold for some Riesz cases [**?**].

To date, this sharp $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$ error rate has only been only proven in the (super-)Coulomb case: first, the Coulomb case [LS18, Ser23, Ros23] up to second order (i.e. $n \le 2$ in (1.4)), and then recently, the entire (super-)Coulomb case at any order [RS24c]. In the present work, we prove a functional inequality with the sharp $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$ error for the sub-Coulomb case at first order. In fact, we prove a sharp functional inequality for all Riesz cases. See the remarks at the end of this subsection for a comparison with the (super-)Coulomb inequality of [RS24c].

In addition to its immediate application to the mean-field limit (see Section 1.4), this sharp functional inequality is crucial for studying fluctuations around the mean-field limit in and out of thermal equilibrium. We use this functional inequality in the forthcoming work [HRS] of J. Huang and the last two authors on quantitative CLTs for the fluctuations of Riesz flows in varying temperature regimes. It also may be incorporated into the transport and Stein's method approaches to CLTs for fluctuations of canonical Gibbs ensembles to establish new results [**?**].

1.2. **New functional inequality.** To state our main result, we define the microscopic length scale

$$\lambda := (N\|\mu\|_{L^\infty})^{-\frac{1}{\mathsf{d}}}, \tag{1.5}$$

which one may regard as the typical inter-particle distance. If one does not wish to track the dependence of $\|\mu\|_{L^\infty}$, one may simply define one can also simply set $\lambda = N^{-1/\mathsf{d}}$ in all of the paper at the cost of letting all constants depend on $\|\mu\|_{L^\infty}$.

Theorem 1.1. *Let $\mathsf{d} \ge 1$, $0 \le \mathsf{s} < \mathsf{d}$, and $\mathsf{a} \in (\mathsf{d}, \mathsf{d}+2)$. There exists a constant $C > 0$ depending only $\mathsf{a}, \mathsf{d}, \mathsf{s}$ such that the following holds. Let $\mu \in L^1(\mathbb{R}^{\mathsf{d}}) \cap L^\infty(\mathbb{R}^{\mathsf{d}})$ with $\int_{\mathbb{R}^{\mathsf{d}}} \mu = 1$. Let $v : \mathbb{R}^{\mathsf{d}} \to \mathbb{R}^{\mathsf{d}}$ be a Lipschitz vector field. For any pairwise distinct configuration $X_N \in (\mathbb{R}^{\mathsf{d}})^N$, it holds that*

$$\begin{aligned} &\left|\int_{(\mathbb{R}^{\mathsf{d}})^2\setminus\triangle} (v(x)-v(y))\cdot\nabla\mathsf{g}(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\Big)^{\otimes 2}(x,y)\right| \\ &\qquad \le C\Big(\|\nabla v\| + \||\nabla|^{\frac{\mathsf{a}}{2}}v\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}}\Big)\Big(\mathsf{F}_N(X_N,\mu) - \frac{\log\lambda}{2N}\mathbf{1}_{\mathsf{s}=0} + C\|\mu\|_{L^\infty}\lambda^{\mathsf{d}-\mathsf{s}}\Big). \end{aligned} \tag{1.6}$$

We note that the additive error term is in $\lambda^{\mathsf{d}-\mathsf{s}} \propto (N^{-1+\frac{\mathsf{s}}{\mathsf{d}}})$, which is the announced optimal estimate. The dependence $\|\mu\|_{L^\infty}$ may be weakened to $\|\mu\|_{L^p}$ for $p > \frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}}$, but at the cost of increasing the size of the additive error term. We defer comments on the regularity assumptions for v until Section 1.5.

In fact, we will obtain Theorem 1.1 as a special case of Theorem 4.1, which establishes an analogous functional inequality valid for a larger class of "Riesz-type" potentials introduced in Section 2.2 (cf. the class of Riesz-type potentials considered in [NRS22]).

Comparing Theorem 1.1 to [NRS22, Proposition 4.1] or [RS23a, Proposition 5.15], one sees that the additive has been improved $\propto N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$; however, the previous transport regularity $\|\nabla v\|_{L^\infty} + \||\nabla|^{\frac{\mathsf{d}-\mathsf{s}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}}} \mathbf{1}_{\mathsf{s}<\mathsf{d}-2}$ has been replaced by $\|\nabla v\|_{L^\infty} + \||\nabla|^{\frac{\mathsf{a}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}}$, for $\mathsf{a} \in (\mathsf{d}, \mathsf{d}+2)$, which, by Sobolev embedding, is stronger. See Section 1.5 for further comments on this point. On the other hand, comparing Theorem 1.1 to [RS24c, Theorem 1.1, (1.7)], which only depends on $\|\nabla v\|_{L^\infty}$, it is evident that the estimate we obtain here in the (super-)Coulomb case is worse. Though, we feel the proof presented here is more elementary than that of [RS24c].

Remark 1.2. Although we only consider the whole space $\mathbb{R}^{\mathsf{d}}$ in this paper, the inequality (and its proof) hold on the flat torus $\mathbb{T}^{\mathsf{d}}$ where g is now taken to be the solution of $|\nabla|^{\mathsf{d}-\mathsf{s}}\mathsf{g} = \mathsf{c}_{\mathsf{d},\mathsf{s}}(\delta_0 - 1)$. We leave the details of this extension to the interested reader.

1.3. **Proof method.** Our method of proof has two main ingredients: (1) a new potential truncation scheme based on an integral representation of the Riesz potential and (2) commutator estimates of Kato-Ponce type.

The starting point for the first ingredient is the representation of the Riesz potential

$$\forall x \neq 0, \qquad \mathsf{g}(x) = \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}} \int_0^\infty t^{\mathsf{d}-\mathsf{s}} \phi_t(x) \frac{dt}{t}, \tag{1.7}$$

where ϕ is any sufficiently nice radial function with nonnegative Fourier transform $\hat{\phi} \geq 0$, $\phi_t(x) := t^{-\mathsf{d}}\phi(x/t)$, and $c_{\phi,\mathsf{d},\mathsf{s}}$ depends on $\phi, \mathsf{d}, \mathsf{s}$. In the log case $\mathsf{s} = 0$, the integral has to be renormalized to obtain a convergent expression due to the fact that $\log|x|$ does not decay at infinity. See Lemma 2.1 for precise assumptions on the validity of the identity. The formula, which is a consequence of scaling, expresses that the Riesz potential is a weighted average of the family of approximate identities $\{\phi_t\}_{t>0}$. In [Rub24, Chapter 6], Rubin refers to this as a "wavelet type representation." The identity (1.7) may also be understood in terms of Mellin transforms. Letting $\mathcal{M}(f)(z)$ denote the Mellin transform of a function f and setting $\psi_{|x|}(t) := \phi(|x|/t)$, we have the relation

$$\int_0^\infty t^{\mathsf{d}-\mathsf{s}} \phi_t(x) \frac{dt}{t} = \int_0^\infty t^{-\mathsf{s}} \phi(|x|/t) \frac{dt}{t} = \mathcal{M}(\psi_{|x|})(-\mathsf{s}) = |x|^{-\mathsf{s}} \mathcal{M}(\phi)(\mathsf{s}). \tag{1.8}$$

The utility of the representation (1.7) is that it conveniently allows us to truncate the singularity of the Riesz potential simply by cutting off the integral for small t. More precisely, given a scale $\eta > 0$, we define the truncated potential

$$\mathsf{g}_\eta(x) := \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}} \int_\eta^\infty t^{\mathsf{d}-\mathsf{s}} \phi_t(x) \frac{dt}{t}. \tag{1.9}$$

While g is singular at the origin, g_η is finite and in fact is bounded by $\mathsf{g}(\eta)$. The truncation has other desirable properties in that it is positive definite (i.e. repulsive), is controlled by g, and the difference $\mathsf{g} - \mathsf{g}_\eta \geq 0$ decays rapidly outside the ball $B(0,\eta)$. We refer to Lemma 2.2 for all of the properties of g_η.

We then show in Proposition 2.11 that the modulated energy $\mathsf{F}_N(X_N, \mu)$ controls the truncated potential energy $\frac{1}{2}\int_{(\mathbb{R}^{\mathsf{d}})^2} \mathsf{g}_\eta(x-y) d(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu)^{\otimes 2}$, which is a genuine energy/maximum mean discrepancy, as well providing control on the interaction energy due to interactions between particles at small scale. The latter such control allows, for instance, to bound in terms of the modulated energy the interaction energy due to nearest neighbors at the microscale $\propto N^{-1/\mathsf{d}}$ (see Corollary 2.13)—a fortiori, bounding the number of points with nearest-neighbor distance below the microscale. Using the potential truncation, we also show with Proposition 2.14 a new coercivity estimate that establishes that the modulated energy controls the squared inhomogeneous Sobolev norm $\|\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\|^2_{H^{-\frac{\mathsf{d}}{2}-}}$ up to $O(\lambda^{\mathsf{d}-\mathsf{s}})$ error.

We emphasize that in contrast to the previous works [PS17, Ser20, NRS22, RS23b], there is no need to consider a smearing/regularizing of the charges δ_x. Throughout the present paper, we will only need to consider the truncated potential g_η.

In previous works on the modulated-energy method, the strategy behind an estimate of the form (1.6) is to first prove an estimate valid when the diagonal is reinserted and $\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu$ is replaced by a more regular distribution f. One then reduces to this regular setting by replacing the point charges δ_{x_i} by smeared charges $\delta_{x_i}^{(\eta_i)}$ and estimating the error from this replacement, the so-called renormalization step. Important to implementing the renormalization is to show that the modulated energy controls both the potential energy of the difference $\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu$ and the small-scale interactions.

The present work differs crucially in two regards. First, rather than regularize $\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu$ through smearing, we regularize g through the truncation g_η.[3] Second, to prove an estimate for g_η (see Proposition 3.1), we use the definition (1.9) to reduce to proving estimates for

$$\int_{(\mathbb{R}^\mathsf{d})^2} (v(x)-v(y))\cdot\nabla\phi(\frac{x-y}{t})d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\Big)^{\otimes 2}(x,y), \qquad t\in(0,\infty), \tag{1.10}$$

which we integrate over (η,∞) with respect to the measure $t^{\mathsf{d}-\mathsf{s}}\frac{dt}{t}$. Provided the dependence on v in our estimate is scale-invariant, this reduces to proving an estimate for $t=1$.

We now come to the second main ingredient of our proof. The formula (1.7) is valid for a large class of scaling functions ϕ, but not all choices of ϕ will allow us to prove the desired estimate. Indeed, it is a elementary Fourier computation that no estimate of the form

$$\int_{(\mathbb{R}^\mathsf{d})^2} (v(x)-v(y))\cdot\nabla\phi(x-y)f(x)f(y) \le C_v\int_{(\mathbb{R}^\mathsf{d})^2}\phi(x-y)f(x)f(y) \tag{1.11}$$

can hold for ϕ Gaussian. It turns out that the a good choice for ϕ is a Bessel potential, i.e. $\hat{\phi}(\xi) = \langle 2\pi\xi\rangle^{-\mathsf{a}} = (1+4\pi^2|\xi|^2)^{-\mathsf{a}/2}$, which is the kernel of the inhomogeneous Fourier multiplier $\langle\nabla\rangle^\mathsf{a}$. The Bessel potential is a screening of the Riesz potential which preserves its local behavior at the origin but avoids the issues at low frequency that lead to the slow decay of the Riesz potential (see Lemma 2.5). Through duality (i.e. setting $h \coloneqq \phi * f$ and using that $\langle\nabla\rangle^\mathsf{a} h = f$) the choice of Bessel potential reduces to proving a product rule for $\langle\nabla\rangle^{\mathsf{a}/2}$. Such product rules are known as Kato-Ponce estimates in the harmonic analysis literature on account of their origins [KP88]. This is the content of Lemma 3.4, our main technical lemma. Here, we rely on commutator estimates of Li [Li19], as well as a local representation of fractional powers $\langle\nabla\rangle^{\mathsf{a}/2}$ via dimension extension (see Lemma 3.5), analogous to the Caffarelli-Silvestre representation for the fractional Laplacian.

The parameter a for the Bessel potential is a degree of freedom, but not one without constraints. We need $\mathsf{a}>\mathsf{d}$, so that ϕ is continuous, bounded. On the other hand, if a is too large, then we will not be able to an estimate whose dependence on v scales properly (see the proof of Lemma 3.4). This leads to the constraint $\mathsf{a}<\mathsf{d}+2$. Even accounting for the constraint on a, it is somewhat remarkable that we obtain a homogeneous estimate in the end through an intermediate inhomogeneous estimate.

With all of the above described ingredients, the conclusion of the proof of is now just a simple matter of bookkeeping.

Returning to a point alluded to in the previous subsection, let us remark that the representation (1.7) itself suggests a class of potentials beyond the exact Riesz case. Namely, we can replace the function $t^{\mathsf{d}-\mathsf{s}}$ by a general function of t which is pointwise controlled above and below by $t^{\mathsf{d}-\mathsf{s}}$. We call these *Riesz-type* potentials and properly introduce them in Section 2.2 (see Definition 2.8).

[3] Remark that the work [BJW19a] also considered a scheme based on regularizing the interaction g at small length scales. However, the procedure in that work is completely different than our own, limited to the spatially periodic setting, and does not yield optimal error estimates, unlike our own procedure.

1.4. **Applications.** We now discuss applications of Theorem 1.1 related to mean-field and supercritical mean-field limits. We will be brief in our remarks, given this subsection closely follows [RS24c, Section 1.4], which has further background.

The first application concerns mean-field limits first-order dynamics of the form

$$\begin{cases} \dot{x}_i^t = \dfrac{1}{N} \displaystyle\sum_{1\leq j\leq N: j\neq i} \mathbb{M}\nabla \mathsf{g}(x_i^t - x_j^t) - \mathsf{V}(x_i^t) \\ x_i^t\big|_{t=0} = x_i^\circ, \end{cases} \qquad i \in [N]. \tag{1.12}$$

Here, $x_i^\circ \in \mathbb{R}^\mathsf{d}$ are the pairwise distinct initial positions, $\mathbb{M}$ is a $\mathsf{d}\times\mathsf{d}$ constant real matrix such that

$$\mathbb{M}\xi\cdot\xi \leq 0 \qquad \forall \xi \in \mathbb{R}^\mathsf{d}, \tag{1.13}$$

which is a repulsivity assumption, and V is an external field (e.g. $-\nabla V$ for some confining potential V). Choosing $\mathbb{M} = -\mathbb{I}$ yields *gradient/dissipative* dynamics, while choosing $\mathbb{M}$ to be antisymmetric yields *Hamiltonian/conservative* dynamics; mixed flows are also permitted. We assume that g is of the form (1.1). One can check that our assumption (1.13) for $\mathbb{M}$ ensures that the energy for (1.12) is nonincreasing, therefore if the particles are initially separated, they remain separated for all time, so that there is a unique, global strong solution to the system (1.12).

The mean-field limit refers to the convergence as $N \to \infty$ of the *empirical measure*

$$\mu_N^t \coloneqq \frac{1}{N}\sum_{i=1}^N \delta_{x_i^t} \tag{1.14}$$

associated to a solution $X_N^t \coloneqq (x_1^t, \ldots, x_N^t)$ of the system (1.12) to a solution μ^t of the Cauchy problem

$$\partial_t \mu = \operatorname{div}((\mathsf{V} - \mathbb{M}\nabla\mathsf{g} * \mu)\mu), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^\mathsf{d}. \tag{1.15}$$

There is a long history to mean-field limits for systems of the form (1.12)—a proper review of which is beyond the scope of this paper. beginning with regular velocities (typically, Lipschitz) [Dob79, Szn91]. Singular interaction are far more challenging and only recently have breakthroughs been made to cover the full Riesz case $\mathsf{s} < \mathsf{d}$: the sub-Coulomb case $\mathsf{s} < \mathsf{d}-2$, [Hau09, CCH14], the Coulomb/super-Coulomb case $\mathsf{d}-2 \leq \mathsf{s} < \mathsf{d}$ [Due16, CFP12, BO19, Ser20], and the full case $0 \leq \mathsf{s} < \mathsf{d}$ [BJW19a, NRS22], thanks in large part to the modulated energy. Further extensions of the modulated energy method have been obtained in the Riesz case in terms of the regularity assumptions on the limiting equation [Ros22b, Ros22a] and incorporating multiplicative [Ros20] and additive noise [RS23a, HC23]. We also mention the relative entropy method [JW18, GBM24, FW23, RS24b], which allows treats $\dot{W}^{-1,\infty}$ forces, essentially corresponding to the $\mathsf{s}=0$ case, and its combination with the modulated energy in the form of the modulated free energy [BJW19b, BJW19a, BJW23, dCRS23b, RS24b], which is well suited to overdamped Langevin dynamics and can even handle logarithmically attractive interactions [BJW19b, BJW23, dCRS23a]. Finally, we mention an exciting recent direction focused on weighted estimates for hierarchies of marginals [Lac23, LLF23, BJS22, Wan24] and cumulants [HCR25, BDJ24]. Although these approaches are currently unable to treat the full Riesz range, they do have advantages in terms of working for both first- and second-order dynamics and can, in certain cases, achieve sharp rates for propagation of chaos (cf. Remark 1.4 below). For a proper discussion of contributions and the techniques behind them, we refer to the recent survey [CD21], the lecture notes [Gol22, Gol16, JW17, Jab14], and the introductions of [Ser20, NRS22].

As explained in Section 1.1, the optimal rate of convergence as $N \to \infty$ of the empirical measure μ_N^t to the solution μ^t of (1.15) in the distance F_N is $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$. To the best of our knowledge, this optimal rate is only known in the (super-)Coulomb case first for stationary solutions, corresponding to minimizers of the associated Coulomb/Riesz energy only results [SS15b, SS15a, RS16, PS17] being for stationary solutions of (1.15) with $\mathbb{M} = -\mathbb{I}$. The sub-Coulomb case is only known for stationary solutions on the flat torus .

Our first application of Theorem 1.1 establishes mean-field convergence at the optimal rate $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$ for the full Riesz case. By the now standard modulated-energy method, the proof is an immediate consequence of Theorem 1.1 (see Section 5.1).

Theorem 1.3. *Let* g *be of the form* (1.1) *and* V *satisfy* $\|\nabla \mathsf{V}\|_{L^\infty} + \||\nabla|^{\frac{\mathsf{a}}{2}}\nabla V\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}} < \infty$ *some* $\mathsf{a} \in (\mathsf{d}, \mathsf{d}+2)$. *Assume the equation* (1.15) *admits a solution* $\mu \in L^\infty([0,T], \mathcal{P}(\mathbb{R}^\mathsf{d}) \cap L^\infty(\mathbb{R}^\mathsf{d}))$, *for some* $T > 0$, *such that for some* $s \in (\mathsf{d}, \mathsf{d}+2)$,

$$\|\nabla^{\otimes 2}\mathsf{g} * \mu^t\|_{L^\infty([0,T],L^\infty)} + \||\nabla|^{\frac{\mathsf{a}}{2}}\nabla \mathsf{g} * \mu^t\|_{L^\infty([0,T],L^{\frac{2\mathsf{d}}{\mathsf{a}-2}})} < \infty. \tag{1.16}$$

If $\mathsf{s} = 0$, *then also assume that* $\int_{\mathbb{R}^\mathsf{d}} \log(1+|x|)d\mu^t(x) < \infty$ *for all* $t \in [0,T]$.[4]

Let X_N *solve* (1.12). *Then there exists a constant* $C > 0$, *depending only on* $|\mathbb{M}|$, d, s, *such that*

$$\begin{aligned} &\mathsf{F}_N(X_N^t, \mu^t) + \frac{\log\big(N\|\mu^t\|_{L^\infty}\big)}{2N\mathsf{d}}\mathbf{1}_{\mathsf{s}=0} + C\|\mu^t\|_{L^\infty}^{\frac{\mathsf{s}}{\mathsf{d}}} N^{\frac{\mathsf{s}}{\mathsf{d}}-1} \\ &\le e^{C\int_0^t(\|\nabla u^\tau\|_{L^\infty} + \||\nabla|^{\frac{s}{2}}u^\tau\|_{L^{\frac{2\mathsf{d}}{s-2}}})d\tau}\left(\mathsf{F}_N(X_N^0,\mu^0) + \sup_{\tau\in[0,t]}\left(\frac{\log(N\|\mu^\tau\|_{L^\infty})}{2N\mathsf{d}}\mathbf{1}_{\mathsf{s}=0} + C\|\mu^\tau\|_{L^\infty}^{\frac{\mathsf{s}}{\mathsf{d}}}N^{\frac{\mathsf{s}}{\mathsf{d}}-1}\right)\right), \end{aligned} \tag{1.17}$$

where $u^t := -\mathbb{M}\nabla \mathsf{g} * \mu^t + \mathsf{V}$.

In particular, if $\mu_N^0 \rightharpoonup \mu^0$ *in the weak-* topology for measures and*

$$\lim_{N\to\infty} F_N(X_N^0, \mu^0) = 0, \tag{1.18}$$

then

$$\mu_N^t \rightharpoonup \mu^t \qquad \forall t \in [0,T]. \tag{1.19}$$

As a consequence of Theorem 1.1, we also have new, improved rates of convergence in the modulated-energy distance for dynamics with noise, compared to the previous work [RS23a] by the last two authors. At the microscopic level, the system (**??**) is generalized to

where $\beta \in (0,\infty]$ has the interpretation of inverse temperature and $W_1, \dots, W_N$ are iid d-dimensional Wiener processes.

Remark 1.4. Theorem 1.3 implies propagation of chaos for the marginals of the system (1.12) with initial data X_N° randomly chosen according to a μ°-chaotic law. For instance, see [Ser20, Remark 3.7] or [RS23a, Remark 1.5] for details. However, this "global-to-local" argument in general leads to a suboptimal rate [Lac23].

Remark 1.5. For any external field of the form $\mathsf{V}(x) = c\Big(-x + \Phi(x)\Big)$, where, say Φ is a Schwartz vector field, the condition $\|\nabla \mathsf{V}\|_{L^\infty} + \||\nabla|^{\frac{\mathsf{a}}{2}}\nabla V\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}} < \infty$ is satisfied. Indeed, it is evident that $\|\nabla\Phi\|_{L^\infty} + \||\nabla|^{\frac{\mathsf{a}}{2}}\nabla\Phi\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}} < \infty$. Since the singular sub-Coulomb case is vacuous for $\mathsf{d} \le 2$, the condition $\mathsf{d} \ge 3$ and $\mathsf{a} \in (\mathsf{d}, \mathsf{d}+2)$ implies that $\frac{\mathsf{s}}{2} > \frac{3}{2}$. But for any $a > 1$, the (vector-valued) distribution $|\nabla|^a x = 0$. The claim now follows by triangle inequality.

The regularity assumption (1.16) for μ^t is a bit more involved. We defer verification until in Section 5.2, where we show that (1.16) is satisfied provided that μ^0 is suitably regular.

[4] This condition is to ensure that the convolution $\mathsf{g} * \mu$ is pointwise defined. None of our estimates will depend on it. Through a Gronwall argument, one checks that if μ^0 satisfies this condition, then μ^t also satisfies this condition uniformly in $[0,T]$.

Remark 1.6. The paper [RS24a] by the last two named authors gave an application of Theorem 1.1 in the (super-)Coulomb case to problem of so-called supercritical mean-field limits, namely the derivation of the Lake equation from the Newtonian N-body problem in the joint mean-field and quasineutral limit. The sharp first-order commutator estimate (1.6) plays an essential role to show convergence provided $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}/\varepsilon^2 \to 0$ as $\varepsilon \to 0$ and $N \to \infty$. This result is sharp, as when $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}/\varepsilon^2 \not\to 0$, we showed through an explicit example that convergence may fail. The present Theorem 1.1 would allow one to extend the results of [RS24a] to the sub-Coulomb case, provided one could also extend the necessary auxiliary results concerning regularity of the boundary of support for equilibrium measures to the sub-Coulomb case. In the periodic setting, where the equilibrium measure takes full support, this last part is not an issue.

1.5. **Remaining questions.** Although we establish a sharp first-order commutator estimate for sub-Coulomb modulated energies, our work raises and leaves open several questions that we briefly discuss below.

The first question is the quantitative dependence on v in the right-hand side of (1.6). In the (super-)Coulomb case, it has been shown [LS18, Ser23, Ros23, RS24c] that the estimate holds with only $\|\nabla v\|_{L^\infty}$. This dependence is essentially sharp, as we show in the forthcoming work [HCRS], in the sense that it cannot be weakened to $\|\nabla v\|_{BMO}$. In the sub-Coulomb case, it has been shown [NRS22] that the estimate holds with $\|\nabla v\|_{L^\infty} + \||\nabla|^{\frac{\mathsf{d}-\mathsf{s}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}}}$. Importantly, both terms scale the same way under the transformation $v \mapsto v(\lambda\cdot)$. Again, this is essentially sharp, as we show in [HCRS] that $\||\nabla|^{\frac{\mathsf{d}-\mathsf{s}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}}}$ cannot be weakened to $\||\nabla|^{a} v\|_{L^{\frac{\mathsf{d}}{a-1}}}$ for any $a < \frac{\mathsf{d}-\mathsf{s}}{2}$. The dependence $\||\nabla|^{\frac{\mathsf{a}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}}$, for any $\mathsf{a} \in (\mathsf{d}, \mathsf{d}+2)$, is an artifact of our passing through an intermediate commutator estimate for Bessel potentials in the proof of Theorem 1.1. By Sobolev embedding, it is stronger than $\||\nabla|^{\frac{\mathsf{d}-\mathsf{s}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}}}$. We expect the latter to be the optimal regularity dependence even with the sharp additive error, but it does not seem obtainable with our current proof.

The second question is the *localizability* of the estimate. To illustrate what we mean, consider the Coulomb case. Then using the so-called electric formulation of the energy, we may set $h_N \coloneqq \mathsf{g} * (\mu_N - \mu)$ and express the modulated energy as a renormalization of $\int_{\mathbb{R}^\mathsf{d}} |\nabla h_N|^2$. This latter quantity may evidently be localized to any domain $\Omega \subset \mathbb{R}^\mathsf{d}$. Letting v be a transport and taking $\Omega = \operatorname{supp} v$, one would like to establish the localized functional inequality

$$\int_{(\mathbb{R}^\mathsf{d})^2} (v(x) - v(y)) \cdot \nabla \mathsf{g}(x-y) d(\mu_N - \mu)^{\otimes 2} \le C_v \int_\Omega |\nabla h_N|^2, \tag{1.20}$$

which is known to hold with $C_v = \|\nabla v\|_{L^\infty}$ [LS18, Ser23]. Such localized estimates are important for applications to CLTs for the fluctuations of Riesz gases at mesocales (e.g. see [Ser23]). An estimate of the form For $= \mathsf{d} - 2\mathsf{k}$ with integer $1 \le k < \frac{\mathsf{d}+2}{2}$, g is the fundamental solution of $(-\Delta)^k$, so it seems natural to consider a localized potential energy $\int_\Omega |\nabla^{\otimes k} h|^2$ and try to show that (1.20) holds with right-hand side now instead $\int_\Omega |\nabla^{\otimes k} h|^2$. For other values s, g is the fundamental solution of a higher-order power of the fractional Laplacian, a nonlocal operator, and the naive quantity $\int_\Omega ||\nabla|^{\frac{\mathsf{d}-\mathsf{s}}{2}} h|^2$ is not a good ansatz for a localized potential energy. Instead, one should probably use the extension procedure for higher powers of the fractional to find a local representation of the energy, analogous to the use of the Caffarelli-Silvestre extension used in the super-Coulomb case [PS17, Ser20] and most recently in [RS24c] to show a sharp localized estimate. Even for the local case $\mathsf{s} = \mathsf{d} - 2k$, the difficulty remains that our proof of the renormalized commutator itself uses nonlocal estimates in the form of these Kato-Ponce commutator bounds, and therefore is not evidently amenable to show the desired localized estimates.

The last question we mention is a sharp estimate for higher-order commutators, which state that for $n \ge 1$, the quantity in (1.4) is again controlled by $C(\mathsf{F}_N(X_N, \mu) + N^{-\alpha})$. Such inequalities were first shown at second order (i.e. $n = 2$) in [LS18, Ros20] for the $\mathsf{d} = 2$ Coulomb case, then in [Ser23] for the general Coulomb case (although with an estimate which is not even optimal in its F_N dependence), and later in [NRS22] for the full Riesz-type case $0 \le \mathsf{s} < \mathsf{d}$. These second-order estimates were important for the same problem of fluctuations of Coulomb gases, as well as for deriving mean-field limits with multiplicative noise. Estimates beyond second order are also useful, allowing to obtain finer estimates on the fluctuations of Riesz gases to treat a broader class of interactions [PS]. Though not explicitly written, the approach of [NRS22] yields higher-order estimates, but which are in general not sharp in their additive error. In contrast, the sharp estimates from [Ser23] are only to second order for the two-dimensional Coulomb case. More importantly, their proof is quite intricate and seems impossible to generalize to higher-order derivatives. In [RS24c], sharp, localized estimates were shown at all orders for the (super-)Coulomb case via a delicate induction argument. In contrast, it is not clear how to extend the method of this paper to $n \ge 2$. A first attempt would be to mimic the strategy of [NRS22]. Namely, write integrate by parts a sufficient number of times to throw off derivatives from the test functions onto the commutator kernel and close with the difference quotient characterization of the Sobolev seminorm ($\mathsf{s} \neq \mathsf{d} - 2k$) or the Christ-Journé theorem for Calderón d-commutators ($\mathsf{s} = \mathsf{d} - 2k$). To be compatible with our potential truncation, we would need to implement this strategy with the Riesz kernel replaced by a Bessel kernel, the inhomogeneity of the latter breaking the argument. A plausible alternative strategy is to desymmetrize (1.4) and write the expression in terms of iterated commutators. But this entails difficulties of considering commutators of commutators, not too mention the algebraic complexity as n increases.

1.6. **Organization of article.** In Section 2, we describe the truncation procedure and its basic properties for Riesz potentials (Section 2.1). We then introduce a class of Riesz-type potentials, alluded to in the introduction, to which we extend the potential truncation (Section 2.2). Finally, we use the truncation scheme to prove optimal lower bounds for the modulated energy and local energy control (Section 2.3), as well as a new, sharp coercivity estimate for the modulated energy (Section 2.4).

In Section 3, we prove some Kato-Ponce type commutator estimates for powers of the inhomogeneous fractional Laplacian. The main result is Proposition 3.1. This is the most technical part of the paper. As a consequence, we have Corollary 3.2, which is the desired functional inequality but for the truncated potential (i.e. pre-renormalization).

In Section 4, we combine the potential truncation of Section 2 with the unrenormalized commutator estimate of Section 3 to prove Theorem 1.1. In fact, we present with Theorem 4.1 a more general functional inequality valid for the Riesz-type potentials introduced in Section 2.2, which includes Theorem 1.1 as a special case.

Finally, in Section 5, we combine Theorem 1.1 with the well-known dissipation relation for the modulated energy to prove Theorem 1.3, the sharp rate of mean-field convergence (Section 5.1). We close the paper by verifying that solutions of the mean-field equation (1.15) have vector fields satisfying the regularity condition of Theorem 1.1 (Section 5.2). Note this does not immediately follow from an L^p assumption on the initial density μ° since for any $\mathsf{a} \in (\mathsf{d}, \mathsf{d}+2)$, $|\nabla|^{\frac{\mathsf{a}}{2}}\nabla \mathsf{g} * \mu = \mathsf{c}_{\mathsf{d},\mathsf{s}} \nabla |\nabla|^{\frac{\mathsf{a}}{2}+\mathsf{s}-\mathsf{d}}\mu$, which cannot be controlled in terms of $\|\mu\|_{L^p}$ in general except when $\mathsf{s} < \frac{\mathsf{d}}{2} - 1$.

1.7. **Notation.** We close the introduction with the basic notation used throughout the article without further comment. We mostly follow the conventions of [NRS22, RS23a, RS23b].

Given nonnegative quantities A and B, we write $A \lesssim B$ if there exists a constant $C > 0$, independent of A and B, such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$. Throughout this paper, C will be used to denote a generic constant which may change from line to line.

$\mathbb{N}$ denotes the natural numbers excluding zero, and $\mathbb{N}_0$ including zero. For $N \in \mathbb{N}$, we abbreviate $[N] \coloneqq \{1, \dots, N\}$. $\mathbb{R}_+$ denotes the positive reals. Given $x \in \mathbb{R}^{\mathsf{d}}$ and $r > 0$, $B(x, r)$ and $\partial B(x, r)$ respectively denote the ball and sphere centered at x of radius r. Given a function f, we denote its support by $\operatorname{supp} f$. The notation $\nabla^{\otimes k} f$ denotes the k-tensor field with components $(\partial_{i_1}\cdots\partial_{i_k} f)_{1\le i_1,\ldots,i_k\le \mathsf{d}}$.

$\mathcal{P}(\mathbb{R}^{\mathsf{d}})$ denotes the space of Borel probability measures on $\mathbb{R}^{\mathsf{d}}$. If μ is absolutely continuous with respect to Lebesgue measure, we shall abuse notation by writing μ for both the measure and its density function. When the measure is clearly understood to be Lebesgue, we shall simply write $\int_{\mathbb{R}^{\mathsf{d}}} f$ instead of $\int_{\mathbb{R}^{\mathsf{d}}} f dx$.

2. The modulated energy

In this section, we discuss properties of the modulated energy functional introduced in (1.3).

2.1. Truncation for Riesz potentials. We first introduce the new scheme for truncating the interaction potential g based on a wavelet-type integral representation of the Riesz potential. This replaces the potential regularization scheme introduced in [NRS22] based on averaging with respect to mollifier. Although this paper focuses on the sub-Coulomb case $\mathsf{s} < \mathsf{d}-2$, we note that the potential truncation works just as well in the Coulomb/super-Coulomb case $\mathsf{d}-2 \le \mathsf{s} < \mathsf{d}$, providing an alternative to the truncation scheme used in [PS17, Ser20, AS21, RS23b] and importantly not requiring a dimension extension for the nonlocal case $\mathsf{d}-2 < \mathsf{s} < \mathsf{d}$.

Lemma 2.1. *Let $\phi : \mathbb{R}^{\mathsf{d}} \to \mathbb{R}$ be radial, continuous, bounded, such that $\phi(r) \to 0$ as $r \to \infty$. Assume that*

$$\int_0^\infty r^{\mathsf{s}-1}|\phi(r)|dr < \infty \ \text{if } \mathsf{s} > 0 \quad \text{and} \quad \int_0^\infty |\log(r)\phi'(r)|dr < \infty \ \text{if } \ \mathsf{s} = 0, \tag{2.1}$$

where we use radial symmetry to commit an abuse of notation. Assume further that if $\mathsf{s} = 0$, then $\lim_{r\to 0^+}(\phi(r) - \phi(0))\log r = 0$. Then setting $\phi_t := t^{-\mathsf{d}}\phi(\cdot/t)$, it holds that

$$\forall x \neq 0, \qquad \mathsf{g}(x) = \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}} \lim_{T\to\infty}\left(\int_0^T t^{\mathsf{d}-\mathsf{s}}\phi_t(x)\frac{dt}{t} - C_{\phi,T}\mathbf{1}_{\mathsf{s}=0}\right), \tag{2.2}$$

where[5]

$$\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}} := \begin{cases} \frac{1}{\mathsf{s}\int_0^\infty r^{\mathsf{s}}\phi(r)\frac{dr}{r}}, & \mathsf{s} > 0 \\ \frac{1}{\phi(0)}, & \mathsf{s} = 0, \end{cases} \tag{2.3}$$

$$C_{\phi,T} := \phi(0)\log(T) - \int_0^\infty \log(r)\phi'(r)dr. \tag{2.4}$$

Proof. When $\mathsf{s} > 0$, clearly the above integral converges whenever $x \neq 0$. On the other hand, making the change of variables $r = |x|/t$ we find that

$$\int_0^\infty t^{-\mathsf{s}}\phi(t^{-1}x)\frac{dt}{t} = \mathsf{s}\mathsf{g}(x)\int_0^\infty r^{\mathsf{s}}\phi(r)\frac{dr}{r}, \tag{2.5}$$

which specifies $\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}$.

When $\mathsf{s} = 0$, letting $r = t^{-1}|x|$ and integrating by parts,

$$\int_0^T t^{-1}\phi(t^{-1}x)dt = \int_{|x|/T}^\infty \phi(r)\frac{dr}{r} = -\phi(|x|/T)\log(|x|/T) - \int_{|x|/T}^\infty \log(r)\phi'(r)dr. \tag{2.6}$$

Letting $T \to \infty$ and using the assumption that $\lim_{r\to 0^+}(\phi(r) - \phi(0))\log r = 0$, we see that (2.2) holds with $c_{\phi,\mathsf{d},0} = \frac{1}{\phi(0)}$. $\square$

[5]Implicitly, we are assuming that the quantities in the denominator defining $\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}$ are nonzero.

The importance of the representation (2.2) is that it conveniently allows to define a truncation of g at small scales simply by truncating the integral for small values of t, and that this truncation well-approximates g, preserves its repulsivity, and is controlled by g. More precisely, for $\eta > 0$, we define the truncated potential at scale η by

$$\mathsf{g}_\eta(x) \coloneqq \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}} \lim_{T\to\infty}\left(\int_\eta^T t^{\mathsf{d}-\mathsf{s}}\phi_t(x)\frac{dt}{t} - C_{T,\phi}\mathbf{1}_{\mathsf{s}=0}\right), \tag{2.7}$$

where $C_{T,\phi}$ is as above. Note that $\mathsf{g} = \mathsf{g}_0$. In the sequel, it will be convenient to introduce the notation

$$\mathsf{f}_\eta \coloneqq \mathsf{g} - \mathsf{g}_\eta, \qquad \mathsf{f}_{\alpha,\eta} \coloneqq \mathsf{g}_\alpha - \mathsf{g}_\eta = \mathsf{f}_\eta - \mathsf{f}_\alpha. \tag{2.8}$$

Evidently, $\mathsf{f}_\eta = \mathsf{f}_{0,\eta}$. The reader may also check that limit on the right-hand side of (2.7) holds not only pointwise but also in the sense of tempered distributions.

The following lemma establishes the key properties of the truncated potential under possible additional assumptions on the scaling function ϕ.

Lemma 2.2. *Let ϕ be as in Lemma 2.1. For $\eta > 0$, the following assertions hold.*

(1)

$$\mathsf{g}_\eta(0) = \begin{cases} \eta^{-\mathsf{s}}\phi(0)\int_0^\infty r^{\mathsf{s}}\phi(r)\frac{dr}{r}, & \mathsf{s} > 0 \\ -\log\eta + \frac{1}{\phi(0)}\int_0^\infty \log r\phi'(r)dr, & \mathsf{s} = 0. \end{cases} \tag{2.9}$$

(2) If $\hat\phi \ge 0$ and $\phi \in L^1$, then for any test function $\varphi \ge 0$ (resp. such that $\varphi(0) = 0$ if $\mathsf{s} = 0$), we have $\langle \widehat{\mathsf{g}_\eta}, \phi\rangle \ge 0$.

(3) If $\phi \ge 0$ and ϕ is decreasing, then $\mathsf{g}_\eta(x) \le C\max(\mathsf{g}(\eta), 1)$. Moreover, if $\mathsf{s} > 0$, then $\mathsf{g}_\eta \ge 0$.

(4) Under the preceding assumptions, $\mathsf{f}_\eta(x) \ge C\eta^{-\mathsf{s}}$ for any $|x| \le \eta$. If $\mathsf{s} = 0$, then there exists $\epsilon > 0$ such that

$$\forall |x| \le \epsilon\eta, \qquad |\mathsf{f}_\eta(x) + \log\left(\frac{|x|}{\eta}\right)| \le C. \tag{2.10}$$

(5) Suppose further that $\phi(x) \le C_\gamma \langle x\rangle^{-\gamma}$ for $\gamma > \mathsf{s}$. Then $0 \le \mathsf{f}_\eta(x) \le C_\gamma \frac{\eta^\gamma}{|x|^{\mathsf{s}+\gamma}}$. Consequently, if $\mathsf{s} > 0$, then $\mathsf{f}_\eta(x) \le C_\gamma \min(\mathsf{g}(x), \frac{\eta^\gamma}{|x|^{\mathsf{s}+\gamma}})$. .

(6) Suppose further that $\phi \in C^1(\mathbb{R}^{\mathsf{d}} \setminus \{0\})$ and there is $c \in (0,1]$ such that $|x||\nabla\phi(x)| \le \phi(cx)$. Then $|x||\nabla\mathsf{f}_\eta(x)| \le C\mathsf{f}_{\eta/c}(x)$.

In all cases, the constant $C > 0$ depends only on $\mathsf{d}, \mathsf{s}, \phi$.

Proof. Starting with (1), observe that

$$\begin{aligned} \mathsf{g}_\eta(0) &= \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}} \lim_{T\to\infty}\left(\phi(0)\int_\eta^T t^{-\mathsf{s}}\frac{dt}{t} - C_{T,\phi}\mathbf{1}_{\mathsf{s}=0}\right) \\ &= \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}} \lim_{T\to\infty}\left(\frac{\phi(0)(T^{-\mathsf{s}} - \eta^{-\mathsf{s}})}{-\mathsf{s}}\mathbf{1}_{\mathsf{s}>0} + \phi(0)\log\left(\frac{T}{\eta}\right)\mathbf{1}_{\mathsf{s}=0} \right. \\ &\qquad \left. - \big(\phi(0)\log(T) + \int_0^\infty \log(y)\phi'(y)dy\big)\mathbf{1}_{\mathsf{s}=0}\right) \\ &= \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\left(\frac{\phi(0)\eta^{-\mathsf{s}}}{\mathsf{s}}\mathbf{1}_{\mathsf{s}>0} + \big(-\phi(0)\log\eta + \int_0^\infty \log r\phi'(r)dr\big)\mathbf{1}_{\mathsf{s}=0}\right). \end{aligned} \tag{2.11}$$

Substituting in the definition (2.3) of $c_{\phi,\mathsf{d},\mathsf{s}}$ then yields (1).

To see (2), observe that Fubini-Tonelli and the identity $\widehat{\phi_t} = \hat{\phi}(t\cdot)$ imply that for any $T > 0$, the Fourier transform

$$\int_{\mathbb{R}^{\mathsf{d}}} e^{-2\pi i x\cdot\xi} \int_\eta^T t^{\mathsf{d}-\mathsf{s}}\phi_t(x)\frac{dt}{t}dx = \int_\eta^T t^{\mathsf{d}-\mathsf{s}}\hat{\phi}(t\xi)\frac{dt}{t}. \tag{2.12}$$

Since $\hat{1} = \delta_0$ in the sense of distributions, it follows from the continuity of the Fourier transform with respect to distributional limits that

$$\widehat{\mathsf{g}_\eta}(\xi) = \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}} \lim_{T\to\infty}\left(\int_\eta^T t^{\mathsf{d}-\mathsf{s}}\hat{\phi}(t\xi)\frac{dt}{t} - C_{T,\phi}\delta_0\mathbf{1}_{\mathsf{s}=0}\right), \tag{2.13}$$

where the equality is understood in the sense of distributions. Evidently, the first term on the right-hand side is positive. Consequently, if $\mathsf{s} > 0$, then for any test function $\varphi \geq 0$, we have $\langle\widehat{\mathsf{g}_\eta}, \varphi\rangle \geq 0$. If $\mathsf{s} = 0$, then the additional hypothesis $\varphi(0) = 0$ kills off the contribution of the Dirac mass.

Now consider (3). If $\phi \geq 0$, then when $\mathsf{s} > 0$, it is evident from the definition (2.7) that $\mathsf{g}_\eta \geq 0$. Trivially bounding $\phi_t \leq t^{-\mathsf{d}}\|\phi\|_{L^\infty}$, we also have

$$\mathsf{g}_\eta(x) = \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_\eta^\infty t^{\mathsf{d}-\mathsf{s}}\phi_t(x)\frac{dt}{t} \leq \|\phi\|_{L^\infty}\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_\eta^\infty t^{-\mathsf{s}}\frac{dt}{t} = \|\phi\|_{L^\infty}\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\mathsf{g}(x). \tag{2.14}$$

When $\mathsf{s} = 0$, note that by making the same change of variable and integrating by parts as in the proof of Lemma 2.1,

$$\begin{aligned}\int_\eta^T t^{\mathsf{d}}\phi_t(x)\frac{dt}{t} - C_{T,\phi} &= \phi(\frac{|x|}{\eta})\log\left(\frac{|x|}{\eta}\right) - \phi(\frac{|x|}{T})\log\left(\frac{|x|}{T}\right) - \int_{|x|/T}^{|x|/\eta}\log(r)\phi'(r)dr - C_{T,\phi}\\ &= \phi(\frac{|x|}{\eta})\log\left(\frac{|x|}{\eta}\right) - \phi(0)\log|x| - \big(\phi(\frac{|x|}{T}) - \phi(0)\big)\log\left(\frac{|x|}{T}\right)\\ &\quad + \int_{|x|/\eta}^\infty \log(r)\phi'(r)dr + \int_0^{|x|/T}\log(r)\phi'(r)dr.\end{aligned} \tag{2.15}$$

Letting $T \to \infty$, the last two terms on the preceding line vanish. Thus,

$$\mathsf{g}_\eta(x) = \frac{\phi(\frac{|x|}{\eta})}{\phi(0)}\log\left(\frac{|x|}{\eta}\right) + \frac{1}{\phi(0)}\int_{|x|/\eta}^\infty \log(r)\phi'(r)dr - \log|x|. \tag{2.16}$$

By assumption (2.1) and that ϕ is decreasing,

$$\frac{1}{\phi(0)}\int_{|x|/\eta}^\infty \log(r)\phi'(r)dr \leq \frac{1}{\phi(0)}\int_{|x|/\eta}^{\min(|x|/\eta,1)} |\log r\phi'(r)|dr < \infty, \tag{2.17}$$

which takes care of the second term on the right-hand side of (2.16). If $|x| \geq \eta$, then the first is $\leq \log\frac{|x|}{\eta}$ by our assumption that ϕ is decreasing. Writing

$$\frac{\phi(\frac{|x|}{\eta})}{\phi(0)}\log\left(\frac{|x|}{\eta}\right) - \log|x| = \frac{\phi(\frac{|x|}{\eta}) - \phi(0)}{\phi(0)}\log\left(\frac{|x|}{\eta}\right) - \log\eta, \tag{2.18}$$

By our assumption that $(\phi(r) - \phi(0))\log r \to 0$ as $r \to 0$, there exists $\epsilon \in (0,1)$ independent of η, such that for $|x| \leq \epsilon\eta$, the first term is ≤ 1. For $\eta \geq |x| \geq \epsilon\eta$, we may crudely bound the magnitude of the first term by $-\log\epsilon$. The desired conclusion now follows from putting together all the preceding cases.

Next, consider (4). Since ϕ is decreasing, for any $|x| \leq \eta$,

$$\mathsf{f}_\eta(x) \geq \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_0^\eta t^{\mathsf{d}-\mathsf{s}}\phi_t(\eta)\frac{dt}{t} = \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\eta^{-\mathsf{s}}\int_1^\infty r^{-\mathsf{s}}\phi(r)\frac{dr}{r}, \tag{2.19}$$

where the final equality follows from the change of variable $r = \eta t^{-1}$. If $\mathsf{s} = 0$, then (2.16) implies that

$$\mathsf{f}_\eta(x) = -\frac{\phi(\frac{|x|}{\eta})}{\phi(0)}\log\left(\frac{|x|}{\eta}\right) - \frac{1}{\phi(0)}\int_{|x|/\eta}^\infty \log(r)\phi'(r)dr. \tag{2.20}$$

Again using that there is an $\epsilon > 0$ such that $\sup_{0<r\le\epsilon}|\phi(r)-\phi(0)||\log r| \le 1$ and triangle inequality, we see that

$$\sup_{|x|\le\epsilon\eta}|\mathsf{f}_\eta(x) + \log\left(\frac{|x|}{\eta}\right)| \le \sup_{0<r\le\epsilon}\frac{|\phi(r)-\phi(0)||\log r|}{\phi(0)} + \frac{1}{\phi(0)}\int_0^\infty |\log r||\phi'(r)|dr < \infty. \tag{2.21}$$

This then completes the proof of (4).

For (5), it is immediate from the definition of g_η and the assumption $\phi \ge 0$ that $\mathsf{f}_\eta \ge 0$. If $\mathsf{s} > 0$, then since $\mathsf{g}_\eta \ge 0$, it also follows that $0 \le \mathsf{f}_\eta(x) \le \mathsf{g}(x)$. Next, majorizing $|\phi_t(x)| \le C_\gamma t^{-\mathsf{d}}\langle x/t\rangle^{-\gamma}$, we have

$$\begin{aligned}\mathsf{f}_\eta(x) = \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_0^\eta t^{\mathsf{d}-\mathsf{s}}\phi_t(x)\frac{dt}{t} &\le C_\gamma \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_0^\eta t^{-\mathsf{s}}\langle |x|/t\rangle^{-\gamma}\frac{dt}{t} \\ &= C_\gamma \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}|x|^{-\mathsf{s}}\int_{\frac{|x|}{\eta}}^\infty r^{\mathsf{s}}\langle r\rangle^{-\gamma}\frac{dr}{r} \\ &\le \frac{C_\gamma \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}}{\gamma-\mathsf{s}}|x|^{-\mathsf{s}}(|x|/\eta)^{\mathsf{s}-\gamma},\end{aligned} \tag{2.22}$$

where we have made the change of variable $r = t^{-1}|x|$ and used that the integral in r converges since $\gamma > \mathsf{s}$ by assumption.

Lastly, for (6), we use the definition (2.7) of g_η to see that

$$|x||\nabla\mathsf{f}_\eta(x)| = |x|\left|\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_0^\eta t^{\mathsf{d}-\mathsf{s}}\nabla\phi_t(x)\frac{dt}{t}\right| \le \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_0^\eta t^{-\mathsf{s}}|t^{-1}x||\nabla\phi(t^{-1}x)|\frac{dt}{t}. \tag{2.23}$$

By our assumption, $|t^{-1}x||\nabla\phi(t^{-1}x)| \le \phi(ct^{-1}x)$ and therefore,

$$\begin{aligned}|x||\nabla\mathsf{f}_\eta(x)| \le \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_0^\eta t^{-\mathsf{s}}\phi(ct^{-1}x)\frac{dt}{t} &= c^{-\mathsf{s}}\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_0^{\eta/c} t^{-\mathsf{s}}\phi(t^{-1}x)\frac{dt}{t} \\ &= c^{-\mathsf{s}}\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\mathsf{f}_{\eta/c}(x),\end{aligned} \tag{2.24}$$

where the penultimate equality follows from the change of variable $t/c \mapsto t$. With the last assertion, the proof of the lemma is complete. $\square$

Remark 2.3. The assumption that ϕ is decreasing is not strictly necessary, but it does simplify the computations. In any case, it is satisfied by the specific choices of ϕ we will later in this paper to obtain Theorem 1.1.

Remark 2.4. The identity (2.13) shows that $\widehat{\mathsf{f}_{\alpha,\eta}} \ge 0$ in the sense of distributions for all $0 \le \mathsf{s} < \mathsf{d}$.

We emphasize the importance of property (5) of f_η to obtain the optimal additive error $\eta^{\mathsf{d}-\mathsf{s}}$ in the sequel. In the previous work [NRS22], the mollification scheme used to define g_η only yields the bound $|\mathsf{f}_\eta(x)| \le C\frac{\eta^2}{|x|^{\mathsf{s}+2}}$. This issue of tails was the main culprit behind the sub-optimality of the error estimates in that work. Note that for the potential truncation scheme used for the (super-)Coulomb case in [PS17, Ser20, RS24c], $\mathsf{f}_\eta(x)$ vanishes when $|x| \ge \eta$, so there is no comparable issue.

So far, we have been unspecific about the choice of ϕ, only requiring that it satisfy certain properties. For later use in Section 4, we will choose ϕ to be a Bessel potential, that is the fundamental solution of the inhomogeneous Fourier multiplier whose symbol is given by $\langle 2\pi\xi\rangle^{\mathsf{a}}$. In the next lemma, we recall some important properties of the Bessel potential. Several assertions of the lemma are already in [Gra14, Proposition 1.2.5]. We include a proof of them anyway in an effort to make the present work self-contained.

Lemma 2.5. *Let $\mathsf{a} > 0$ and $G_{\mathsf{a}}(x)$ be defined by $\widehat{G_{\mathsf{a}}}(\xi) = (1+4\pi^2|\xi|^2)^{-\mathsf{a}/2}$. Then $G_{\mathsf{a}} \in L^1(\mathbb{R}^{\mathsf{d}}) \cap C^\infty(\mathbb{R}^{\mathsf{d}} \setminus \{0\})$ is strictly positive and decreasing. Moreover, for any multi-index $\vec{\alpha} \in \mathbb{N}_0^{\mathsf{d}}$, we have for any $\epsilon > 0$,*

$$\forall |x| \geq 2, \qquad |\partial_{\vec{\alpha}} G_{\mathsf{a}}(x)| \leq C_\epsilon e^{-\frac{|x|}{4+\epsilon}} \tag{2.25}$$

and

$$\forall |x| \leq 2, \tag{2.26}$$

Furthermore, for any $\epsilon \in (0,1)$

$$\forall |x| > 0, \qquad |x|^{|\vec{\alpha}|}|\partial_{\vec{\alpha}} G_{\mathsf{a}}(x)| \leq C_\epsilon G_{\mathsf{a}}(\epsilon x) \tag{2.27}$$

Above, $C, C_\epsilon > 0$ depend on $\mathsf{d}, \mathsf{a}, |\vec{\alpha}|$.

Proof. Using the gamma function identity $A^{-\frac{s}{2}} = \frac{1}{\Gamma(s/2)}\int_0^\infty e^{-tA}t^{s/2}\frac{dt}{t}$, we see that

$$(1+4\pi^2|\xi|^2)^{-s/2} = \frac{1}{\Gamma(s/2)}\int_0^\infty e^{-\pi|2\sqrt{t\pi}\xi|^2}t^{s/2}\frac{dt}{t}. \tag{2.28}$$

Taking inverse Fourier transforms of both sides,

$$G_{\mathsf{a}}(x) = \frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)}\int_0^\infty e^{-t}e^{-|x|^2/4t}t^{\frac{s-\mathsf{d}}{2}}\frac{dt}{t}. \tag{2.29}$$

It follows immediately from this identity that $G_{\mathsf{a}} > 0$, smooth on $\mathbb{R}^{\mathsf{d}}$, and $\int_{\mathbb{R}^{\mathsf{d}}} G_{\mathsf{a}} = 1$.

Suppose that $|x| \geq 2$. Let $\vec{\alpha} \in \mathbb{N}_0^{\mathsf{d}}$ be a multi-index. Differentiating inside the integral,

$$\partial_{\vec{\alpha}} G_{\mathsf{a}}(x) = \frac{(-2)^{|\vec{\alpha}|}}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)}\int_0^\infty \prod_{i=1}^{\mathsf{d}} H_{\alpha_i}(x_i/2\sqrt{t})e^{-t}e^{-|x|^2/4t}t^{\frac{s+|\vec{\alpha}|-\mathsf{d}}{2}}\frac{dt}{t}, \tag{2.30}$$

where H_n is the n-th (physicist's) Hermite polynomial. Since $|H_n(y)| \leq C_n(1+|y|^n)$ and

$$t + \frac{|x|^2}{ct} \geq \max(\sqrt{1/c}|x|, t + \frac{4}{ct}) \Longrightarrow t + \frac{|x|^2}{ct} \geq \frac{|x|}{2\sqrt{c}} + \frac{t}{2} + \frac{2}{ct}, \tag{2.31}$$

it follows that for any $c > 4$,

$$|\partial_{\vec{\alpha}} G_{\mathsf{a}}(x)| \leq \frac{C_{|\vec{\alpha}|}}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)}e^{-\frac{|x|}{2\sqrt{c}}}\int_0^\infty e^{-t/2}e^{-2/(ct)}t^{\frac{s+|\vec{\alpha}|-\mathsf{d}}{2}}\frac{dt}{t}. \tag{2.32}$$

Evidently, the integral in t converges.

For $|x| \leq 2$, we decompose

$$G_{\mathsf{a}} = G_{s,1} + G_{s,2} + G_{s,3}, \tag{2.33}$$

where

$$G_{s,1}(x) \coloneqq |x|^{s-\mathsf{d}}\frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)}\int_0^1 e^{-t|x|^2}e^{-1/4t}t^{\frac{s-\mathsf{d}}{2}}\frac{dt}{t}, \tag{2.34}$$

$$G_{s,2}(x) \coloneqq \frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)}\int_{|x|^2}^4 e^{-t}e^{-|x|^2/4t}t^{\frac{s-\mathsf{d}}{2}}\frac{dt}{t}, \tag{2.35}$$

$$G_{s,3}(x) \coloneqq \frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)}\int_4^\infty e^{-t}e^{-|x|^2/4t}t^{\frac{s-\mathsf{d}}{2}}\frac{dt}{t}. \tag{2.36}$$

By the Leibniz rule,

$$\partial_{\vec{\alpha}} G_{s,1}(x) = \sum_{\vec{\beta}\le\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}} \partial_{\vec{\beta}}(|x|^{s-\mathsf{d}}) \frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \int_0^1 \partial_{\vec{\alpha}-\vec{\beta}}(e^{-t|x|^2}) e^{-1/4t} t^{\frac{s-\mathsf{d}}{2}} \frac{dt}{t}$$

$$= \sum_{\vec{\beta}\le\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}} \partial_{\vec{\beta}}(|x|^{s-\mathsf{d}}) \frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \int_0^1 (-\sqrt{t})^{|\vec{\alpha}-\vec{\beta}|} \prod_{i=1}^{\mathsf{d}} H_{\alpha_i-\beta_i}(\sqrt{t}x_i) e^{-t|x|^2} e^{-1/4t} t^{\frac{s-\mathsf{d}}{2}} \frac{dt}{t}. \tag{2.37}$$

If $\vec{\beta}\neq\vec{\alpha}$, then we may bound $|\partial_{\vec{\beta}}|x|^{s-\mathsf{d}}| \le C|x|^{s-|\vec{\beta}|-\mathsf{d}}$, $|\sqrt{t}x|\le 2$, and $e^{-t|x|^2}\le 1$, to obtain

$$\left|\partial_{\vec{\beta}}(|x|^{s-\mathsf{d}}) \frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \int_0^1 (-\sqrt{t})^{|\vec{\alpha}-\vec{\beta}|} \prod_{i=1}^{\mathsf{d}} H_{\alpha_i-\beta_i}(\sqrt{t}x_i) e^{-t|x|^2} e^{-1/4t} t^{\frac{s-\mathsf{d}}{2}} \frac{dt}{t}\right| \le \frac{C_{|\vec{\alpha}|}}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} |x|^{s-\mathsf{d}-|\vec{\beta}|} \int_0^1 e^{-1/4t} t^{\frac{s-\mathsf{d}}{2}} \frac{dt}{t}, \tag{2.38}$$

where the integral in t evidently converges. If $\vec{\beta}=\vec{\alpha}$, then Taylor expanding $e^{-t|x|^2} = 1+O(t|x|^2)$, it follows that

$$\left|\frac{\partial_{\vec{\alpha}}(|x|^{s-\mathsf{d}})}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \int_0^1 \left(e^{-t|x|^2}-1\right) e^{-1/4t} t^{\frac{s-\mathsf{d}}{2}} \frac{dt}{t}\right| \le \frac{C_{|\vec{\alpha}|}|x|^{s+2-|\vec{\alpha}|-\mathsf{d}}}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \int_0^1 e^{-1/4t} t^{\frac{s-\mathsf{d}}{2}} dt. \tag{2.39}$$

We conclude that

$$\left|\frac{\partial_{\vec{\alpha}} G_{s,1}(x)}{\partial_{\vec{\alpha}}|x|^{s-\mathsf{d}}}\right| \le C, \tag{2.40}$$

where $C = C(\mathsf{d}, s, |\vec{\alpha}|)$. Letting $\vec{\alpha}_i := \vec{\alpha} - \vec{e}_i$, we have by the Leibniz rule,

$$\partial_{\vec{\alpha}} G_{s,2}(x) = \frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \sum_{i:\alpha_i>0} \alpha_i \partial_{\vec{\alpha}_i}\Big((-2x_i)e^{-|x|^2} e^{-1/4}|x|^{s-\mathsf{d}-2}\Big) + \frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \int_{|x|^2}^4 e^{-t}\partial_{\vec{\alpha}}\Big(e^{-|x|^2/4t}\Big) t^{\frac{s-\mathsf{d}}{2}} \frac{dt}{t}. \tag{2.41}$$

Lastly, using that $0 \le \frac{|x|^2}{4t} \le \frac14$, we have that

$$|\partial_{\vec{\alpha}} G_{s,3}(x)| \le \frac{1}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \int_4^\infty (2\sqrt{t})^{-|\vec{\alpha}|} \prod_{i=1}^{\mathsf{d}} |H_{\alpha_i}(x_i/2\sqrt{t})| e^{-t} e^{-|x|^2/4t} t^{\frac{s-\mathsf{d}}{2}} \frac{dt}{t}$$

$$\le \frac{C_{|\vec{\alpha}|}}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \int_4^\infty t^{\frac{s-|\vec{\alpha}|-\mathsf{d}}{2}} e^{-t}\frac{dt}{t}. \tag{2.42}$$

Finally, recalling the identity (2.30), majorizing $|H_{\alpha_i}(y)| \le C_{\alpha_i}(1+|y|)^{\alpha_i}$, and using that for any $c<1$, there is a constant $C_{|\vec{\alpha}|,c}>0$ such that $(1+|y|)^{|\vec{\alpha}|}e^{-|y|} \le C_{|\vec{\alpha}|,c}e^{-c|y|}$, we obtain that

$$|x|^{|\vec{\alpha}|}|\partial_{\vec{\alpha}} G_{\mathsf{a}}(x)| \le C_{|\vec{\alpha}|,\epsilon} \frac{2^{|\vec{\alpha}|}}{(2\sqrt{\pi})^{\mathsf{d}}\Gamma(s/2)} \int_0^\infty e^{-t} e^{-|x|^2/(4+\epsilon)t} t^{\frac{s+|\vec{\alpha}|-\mathsf{d}}{2}} \frac{dt}{t}$$

$$= 2^{|\vec{\alpha}|} C_{|\vec{\alpha}|,\epsilon} G_s(\frac{4}{4+\epsilon}x), \tag{2.43}$$

for any $\epsilon>0$. This establishes (2.27) and completes the proof of the lemma.

∎

Remark 2.6. An examination of the $|x| \le 2$ part of the proof of Lemma 2.5 reveals that if $s > \mathsf{d}$, then the matrix field $x \otimes \nabla G_{\mathsf{a}}(x)$ has a $(\mathsf{a}-\mathsf{d}-\epsilon)$-Hölder continuous, for any $\epsilon > 0$, extension that vanishes at $x = 0$. As

$$\begin{aligned}|x \otimes \nabla \mathsf{g}_\eta(x) - y \otimes \nabla \mathsf{g}_\eta(y)| &= \left|\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}} \int_\eta^\infty t^{-\mathsf{s}-2}\Big(x \otimes \nabla G_{\mathsf{a}}(x/t) - y \otimes \nabla G_{\mathsf{a}}(y/t) dt\right| \\ &\le C_\epsilon \int_\eta^\infty t^{-\mathsf{s}-1}|(x-y)/t|^{\mathsf{a}-\mathsf{d}-\epsilon} dt \\ &= C_{\epsilon,\mathsf{a},\mathsf{s},\mathsf{d}}|x-y|^{\mathsf{a}-\mathsf{d}-\epsilon},\end{aligned} \tag{2.44}$$

it follows that $x \otimes \nabla \mathsf{g}_\eta$ also admits an $(s - \mathsf{d} - \epsilon)$-Hölder continuous extension that vanishes at the origin, for any $\eta > 0$.

Remark 2.7. In particular, Lemma 2.5 shows that assertions (2)-(4) of Lemma 2.2 are satisfied when $\phi = G_{\mathsf{a}}$ for $\mathsf{a} > \mathsf{d}$. Furthermore, assertion (5) holds for any $\gamma > \mathsf{s}$ and assertion (6) holds for $c = \frac{1}{2}$.

2.2. **Truncation for Riesz-type potentials.** The identity (2.2) and the fact the choice $\phi = G_{\mathsf{a}}$ satisfies all the assumptions of Lemma 2.2 motivates our considering a class of *Riesz-type* potentials, mentioned in the introduction, that have analogous properties to the exact Riesz potential.

*** Changed around structure of definition. ***

Definition 2.8. Let $\phi : \mathbb{R}^{\mathsf{d}} \to [0, \infty)$ satisfy all the conditions imposed in Lemmas 2.1 and 2.2.

We say that a potential $\mathsf{g} : \mathbb{R}^{\mathsf{d}} \setminus \{0\} \to \mathbb{R}$ is (s, ϕ)*-admissible* if g admits the representation

$$\forall x \neq 0, \qquad \mathsf{g}(x) = \lim_{T\to\infty} \left(\int_0^T \zeta(t)\phi_t(x)\frac{dt}{t} - C_{\phi,T}\mathbf{1}_{\mathsf{s}=0}\right), \tag{2.45}$$

where $C_{\phi,T}$ is as in (2.4), $\zeta : (0, \infty) \to [0, \infty)$ is a locally integrable function such that for some $C_\zeta > 0$,

$$\forall t > 0, \qquad C_\zeta^{-1} t^{\mathsf{d}-\mathsf{s}} \le \zeta(t) \le C_\zeta t^{\mathsf{d}-\mathsf{s}}, \tag{2.46}$$

and if $\mathsf{s} = 0$, then there is a locally integrable function $\rho : (0, \infty) \to [0, \infty)$ such that

$$\zeta(t) = t^{\mathsf{d}} + \rho(t), \qquad \text{where} \int_0^\infty \rho(t)t^{-\mathsf{d}}\frac{dt}{t} < \infty. \tag{2.47}$$

When $\phi = G_{\mathsf{a}}$ for some $\mathsf{a} > 0$, we will just say that g is (s, a)*-admissible*.

To the best of our knowledge, this class of admissible potentials has not been considered in the literature at this level of generality. We note that when ϕ is Gaussian, our definition coincides with a special case of so-called *G-type* potentials considered in [BHS19, Definition 10.5.] and related works. The work [NRS22] by the last two authors and Q.H. Nguyen considered a class of Riez-type potentials that satisfy a superharmonicity condition in a ball around the origin as well as satisfying pointwise physical and Fourier bounds comparable to an exact-Riesz potential. In the super-Coulomb case, similar assumptions are imposed but in a dimension-extended space, viewing the potential g as the restriction of some G to a lower-dimensional subspace. Similarly, the reader may check from Definition 2.8 that an (s, ϕ)-admissible potential, for $\mathsf{s} > 0$, satisfies pointwise bounds in physical and Fourier space comparable to an exact Riesz potential. In the case $\mathsf{s} = 0$, an (s, ϕ)-admissible potential agrees with the $-\log|x|$ up to a more regular, positive-definite remainder. It is not clear that one definition is a proper subset of the other. In particular, the role of superharmonicity here is not apparent as it is in [NRS22]. Although the class in [NRS22] feels broader, the generalization in the present work is more straightforward.

Let $\eta > 0$. If $\mathsf{s} > 0$, then we define g_η just as in (2.7). If $\mathsf{s} = 0$, then we only need to truncate the contribution from $c_\zeta t^{\mathsf{d}}$ in the decomposition for ζ. This is because the assumption $\int_0^\infty \rho(t)t^{-\mathsf{d}}\frac{dt}{t} < \infty$ ensures that $\int_0^\infty \rho(t)\phi_t(x)\frac{dt}{t}$ converges any $x \in \mathbb{R}^{\mathsf{d}}$. More precisely, we define

$$\mathsf{g}_\eta(x) := \begin{cases} \displaystyle\int_\eta^\infty \zeta(t)\phi_t(x)\frac{dt}{t}, & \mathsf{s} \neq 0 \\ \displaystyle\lim_{T\to\infty}\left(\int_\eta^T t^{\mathsf{d}}\phi_t(x)\frac{dt}{t} + \int_0^\infty \rho(t)\phi_t(x)\frac{dt}{t} - C_{T,\phi}\mathbf{1}_{\mathsf{s}=0}\right), & \mathsf{s} = 0. \end{cases} \tag{2.48}$$

For comparison with the exact Riesz potential considered in the previous subsection, it will be convenient to introduce the distinguished notation

$$\mathsf{g}_{\mathrm{Riesz}}(x) := \begin{cases} \frac{1}{\mathsf{s}}|x|^{-\mathsf{s}}, & \mathsf{s} \neq 0 \\ -\log|x|, & \mathsf{s} = 0 \end{cases}. \tag{2.49}$$

Similarly, we write $\mathsf{g}_{\mathrm{Riesz},\eta}, \mathsf{f}_{\mathrm{Riesz},\eta}$ to denote the exact Riesz case.

Analogous to Lemma 2.2, we have the following proposition, which is the main result of this subsection and underlies the results of Sections 2 and 4.

Proposition 2.9. *Let g be (s,ϕ)-admissible. The following assertions hold.*

(1) If $\mathsf{s} > 0$, then $\mathsf{g}_\eta(0) \cong \mathsf{g}_{\mathrm{Riesz}}(\eta)$; and if $\mathsf{s} = 0$, then $|\mathsf{g}_\eta(0) - \mathsf{g}_{\mathrm{Riesz}}(\eta)| \leq C$. In both cases, C depends on ζ and also on ρ if $\mathsf{s} = 0$.
(2) For any test function $\varphi \geq 0$ (resp. such that $\varphi(0) = 0$ if $\mathsf{s} = 0$), we have $\langle \widehat{\mathsf{g}_\eta}, \phi\rangle \geq 0$.
(3) $\mathsf{g}_\eta(x) \leq C\max(\mathsf{g}(\eta), 1)$ and if $\mathsf{s} > 0$, then $\mathsf{g}_\eta \geq 0$.
(4) $\mathsf{f}_\eta(x) \geq C\eta^{-\mathsf{s}}$ for any $|x| \leq \eta$.
(5) For any $\gamma > \mathsf{s}$, we have $0 \leq \mathsf{f}_\eta(x) \leq C_\gamma \min(\mathsf{g}(x), \frac{\eta^\gamma}{|x|^{\mathsf{s}+\gamma}})$.
(6) $|x||\nabla \mathsf{f}_\eta(x)| \leq C\mathsf{f}_{2\eta}(x)$.

In all cases, the constant $C > 0$ depends only on $\mathsf{d}, \mathsf{s}, \zeta$. In particular, all the assertions hold if g is (s, a)-admissible for $\mathsf{a} > \mathsf{d}$.

Proof. The assertions essentially follow from the same reasoning as in the proof of Lemma 2.2. We briefly sketch the details.

Consider first the case $\mathsf{s} > 0$. Then by the assumption (2.46) for ζ, it follows that

$$C^{-1}\mathsf{g}_{\mathrm{Riesz},\eta}(x) = C^{-1}\int_\eta^\infty \zeta(t)\phi_t(x)\frac{dt}{t} \leq \mathsf{g}_\eta(x) = \int_\eta^\infty \zeta(t)\phi_t(x)\frac{dt}{t} \leq C\int_\eta^\infty t^{\mathsf{d}-\mathsf{s}}\phi_t(x)\frac{dt}{t} = C\mathsf{g}_{\mathrm{Riesz},\eta}(x). \tag{2.50}$$

Similarly,

$$C^{-1}\mathsf{f}_{\mathrm{Riesz},\eta}(x) = C^{-1}\int_0^\eta t^{\mathsf{d}-\mathsf{s}}\phi_t(x)\frac{dt}{t} \leq \mathsf{f}_\eta(x) = \int_0^\eta \zeta(t)\phi_t(x)\frac{dt}{t} \leq C\int_0^\eta t^{\mathsf{d}-\mathsf{s}}\phi_t(x)\frac{dt}{t} = C\mathsf{f}_{\mathrm{Riesz},\eta}(x). \tag{2.51}$$

We can now appeal to the corresponding assertions for $\mathsf{g}_{\mathrm{Riesz},\eta}, \mathsf{f}_{\mathrm{Riesz},\eta}$ from Lemma 2.2 to conclude the proposition in the case $\mathsf{s} > 0$, with the exception of assertion (6). For this, we may simply bound

$$|x||\nabla \mathsf{f}_\eta(x)| \leq \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_0^\eta \zeta(t)|x||\nabla\phi_t(x)|\frac{dt}{t} \leq C\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_0^\eta t^{-\mathsf{s}}|t^{-1}x||\nabla\phi(t^{-1}x)|\frac{dt}{t}. \tag{2.52}$$

We can now proceed with no changes as in the proof of Lemma 2.2(6) to obtain the preceding right-hand is $\leq$

$$C'c^{-\mathsf{s}}\mathsf{f}_{\mathrm{Riesz},\eta/c}(x) \leq \frac{C'}{C}c^{-\mathsf{s}}\mathsf{f}_{\eta/c}(x). \tag{2.53}$$

The desired conclusion now follows.

For the case $\mathsf{s} = 0$, we observe from assumption (2.47) and the definition (2.48) of g_η that

$$\mathsf{g}_\eta(x) - \mathsf{g}_{\mathrm{Riesz},\eta}(x) = \int_0^\infty \rho(t)\phi_t(x)\frac{dt}{t}, \tag{2.54}$$

$$\mathsf{f}_\eta(x) = \int_0^\eta t^{\mathsf{d}}\phi_t(x)\frac{dt}{t} = \mathsf{f}_{\mathrm{Riesz},\eta}(x). \tag{2.55}$$

The relation (2.54) gives assertions (1), (2), (3); and the relation (2.55) yields the assertions (4), (5), (6) for f_η. This then completes the proof of the proposition.

□

Moreover, by Lemma 2.2(5) and Hölder's inequality, we have the useful convolution bounds, which will be used in Section 2.3 to control the additive errors.

Lemma 2.10. *Let $p \in (\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}}, p]$. There exists $C > 0$ depending quantitatively only on $\mathsf{d}, \mathsf{s}, p, \phi, \zeta$, such that for $\eta \geq 0$,*

$$\|\mathsf{f}_\eta * \mu\|_{L^\infty} \leq C\|\mu\|_{L^p}\eta^{\frac{\mathsf{d}(p-1)}{p}-\mathsf{s}}, \tag{2.56}$$

$$\||\cdot||\nabla\mathsf{f}_\eta| * \mu\|_{L^\infty} \leq C\|\mu\|_{L^p}\eta^{\frac{\mathsf{d}(p-1)}{p}-\mathsf{s}}. \tag{2.57}$$

Proof. For $x \in \mathbb{R}^{\mathsf{d}}$, we split

$$\begin{aligned}
\int_{\mathbb{R}^{\mathsf{d}}} \mathsf{f}_\eta(x-y)d\mu(y) &= \int_{|x-y|\leq\eta/2} \mathsf{f}_\eta(x-y)d\mu(y) + \int_{|x-y|>\eta/2} \mathsf{f}_\eta(x-y)d\mu(y) \\
&\leq C\eta^{-\mathsf{s}}\int_{|x-y|\leq\eta/2} \mathsf{g}(\frac{x-y}{\eta})d\mu(y) \\
&\quad + C\int_{|x-y|>\eta/2} |x-y|^{-\mathsf{s}}(|x-y|/\eta)^{\mathsf{s}-\gamma}d\mu(y) \\
&\leq C'\Bigg(\eta^{\frac{\mathsf{d}(p_1-1)}{p_1}-\mathsf{s}} + \eta^{\frac{\mathsf{d}(p_1-1)}{p_1}}\mathbf{1}_{\mathsf{s}=0}\Big(\int_0^1 |\log r|^{\frac{p_1}{p_1-1}}dr\Big)^{\frac{p_1-1}{p_1}}\Bigg)\|\mu\|_{L^{p_1}} \\
&\quad + C'\eta^{\frac{\mathsf{d}(p_2-1)}{p_2}-\mathsf{s}}\Big(\int_1^\infty r^{-\frac{\mathsf{s}p_2}{p_2-1}}r^{\frac{p_2(\mathsf{s}-\gamma)}{p_2-1}}dr\Big)^{\frac{p_2-1}{p_2}}\|\mu\|_{L^{p_2}} \\
&= C''\Big(\|\mu\|_{L^{p_1}}\eta^{\frac{\mathsf{d}(p_1-1)}{p_1}-\mathsf{s}} + \|\mu\|_{L^{p_2}}\eta^{\frac{\mathsf{d}(p_2-1)}{p_2}-\mathsf{s}}\Big),
\end{aligned} \tag{2.58}$$

where we have used Proposition 2.9(5) and the applications of Hölder's inequality are valid provided that $\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}} < p_1 \leq \infty$ and $1 \leq p_2 < \frac{\mathsf{d}}{\mathsf{d}-\gamma}$, with the convention that $p_2 = \infty$ is allowed if $\gamma > \mathsf{d}$. In particular, if $\gamma > \mathsf{d}$, then we may take $p_1 = p_2$. Similarly, using Proposition 2.9(6) and then applying the previous estimate (2.56) (assuming $\eta \leq \frac{c}{2}$), for any $x \in \mathbb{R}^{\mathsf{d}}$,

$$\int_{\mathbb{R}^{\mathsf{d}}} |x-y||\nabla\mathsf{f}_\eta(x-y)|d\mu(y) \leq C\int_{\mathbb{R}^{\mathsf{d}}} \mathsf{f}_{\eta/c}(x-y)d\mu(y) \leq C'\|\mu\|_{L^p}\eta^{\frac{\mathsf{d}(p-1)}{p}-\mathsf{s}}, \tag{2.59}$$

again under the same assumptions on p. □

2.3. **Energy control.** Throughout this subsection, we assume that g is (s,ϕ)-admissible where the function ϕ satisfies all the assumptions imposed in Lemmas 2.1 and 2.2.

When $0 < \mathsf{s} < \mathsf{d}$, then it is immediate from basic potential theory (e.g. see [RS23a, Remark 2.5]) that $\mathsf{g} * \mu$ is a bounded, continuous function (it is actually $C^{k,\alpha}$ for some $k \in \mathbb{N}_0$ and $\alpha > 0$ depending on the value of s) and therefore the modulated energy is well-defined. If $\mathsf{s} = 0$, then we need to impose a suitable decay assumption on μ to compensate for the logarithmic growth of g at infinity. The energy condition $\int_{(\mathbb{R}^{\mathsf{d}})^2} \mathsf{g}(x-y)d|\mu|^{\otimes 2}(x,y) < \infty$ from the statement of Theorem 1.1 suffices.

Our main proposition is that the modulated energy controls both the truncated Riesz energy of $\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu$ and the difference between the microscopic energies associated to g and g_η, up to an $O(\eta^{\mathsf{d}-\mathsf{s}})$ additive error. As previously remarked, the size of this error is in general optimal. The reader should compare this proposition to [NRS22, Proposition 2.1] in the sub-Coulomb case and [RS24c, Proposition 2.1] in the super-Coulomb case.

Proposition 2.11. *Let $\mu \in L^1(\mathbb{R}^\mathsf{d}) \cap L^p(\mathbb{R}^\mathsf{d})$ for $\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}} < p \le \infty$ with $\int_{\mathbb{R}^\mathsf{d}} \mu = 1$. Suppose further that $\int_{(\mathbb{R}^\mathsf{d})^2} \mathsf{g}(x-y) d\mu^{\otimes 2}(x,y) < \infty$ and $\int_{(\mathbb{R}^\mathsf{d})^2} |\mathsf{g}|(x-y) d\mu^{\otimes 2}(x,y) < \infty$ if $\mathsf{s} = 0$. Let $X_N \in (\mathbb{R}^\mathsf{d})^N$ be a pairwise distinct configuration. Then*

(2.60)
$$\frac{1}{2N^2}\sum_{1\le i\ne j\le N} \mathsf{f}_\eta(x_i-x_j) + \frac{1}{2}\int_{(\mathbb{R}^\mathsf{d})^2} \mathsf{f}_\eta(x-y)d\mu^{\otimes 2}(x,y) + \frac{1}{2}\int_{(\mathbb{R}^\mathsf{d})^2} \mathsf{g}_\eta(x-y) d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2} \\ \le \mathsf{F}_N(X_N,\mu) + \frac{(\mathsf{g}_{\mathrm{Riesz}}(\eta) + C_\rho)}{2N}\mathbf{1}_{\mathsf{s}=0} + C_\zeta \frac{\mathsf{g}_{\mathrm{Riesz}}(\eta)}{2N}\mathbf{1}_{\mathsf{s}>0} + C\|\mu\|_{L^p}\eta^{\frac{\mathsf{d}(p-1)}{p}-\mathsf{s}},$$

where $C>0$ depends only on $\mathsf{d}, \mathsf{s}, p, \zeta, \phi$ and $C_\rho, C_\zeta \ge 0$, where $C_\rho = 0$ and $C_\zeta = 1$ in the exact Riesz case.

Proof. Unpacking the definition (1.3) of $\mathsf{F}_N(X_N,\mu)$, inserting the identity $\mathsf{g} = \mathsf{f}_\eta + \mathsf{g}_\eta$, and then expanding, we find that

$$\mathsf{F}_N(X_N,\mu) = \frac{1}{2N^2}\sum_{1\le i\ne j\le N}\mathsf{f}_\eta(x_i-x_j) + \frac{1}{2}\int_{(\mathbb{R}^\mathsf{d})^2}\mathsf{f}_\eta(x-y)d\mu^{\otimes 2}(x,y) \\ + \frac{1}{2}\int_{(\mathbb{R}^\mathsf{d})^2}\mathsf{g}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i} - \mu\Big)^{\otimes 2}(x,y) - \frac{1}{2N}\mathsf{g}_\eta(0) + \frac{1}{N}\sum_{i=1}^N\int_{\mathbb{R}^\mathsf{d}}\mathsf{f}_\eta(x-x_i)d\mu(x). \tag{2.61}$$

Applying the estimate (2.56) to the ultimate term and Proposition 2.9(1) to the penultimate term on preceding the right-hand side , the desired conclusion follows. ☐

Remark 2.12. In particular, for the exact Riesz case, choosing $\eta = \lambda = (N\|\mu\|_{L^p})^{-\frac{p}{\mathsf{d}(p-1)}}$, we find the lower bound

$$\mathsf{F}_N(X_N,\mu) - \frac{\log\lambda}{2N}\mathbf{1}_{\mathsf{s}=0} + C\|\mu\|_{L^p}\lambda^{\mathsf{d}-\mathsf{s}} \ge 0, \tag{2.62}$$

where $C = C(\mathsf{d},\mathsf{s}) > 0$, showing the almost positivity of the modulated energy with the optimal size of additive error when $p=\infty$.[6]

As a corollary of Proposition 2.11, we control the small-scale interaction (cf. [Ser23, Corollary 3.4], [RS24c, Proposition 2.3]).

Corollary 2.13. *Introduce the nearest-neighbor type distance*

$$\mathsf{r}_i := \frac{1}{4}\min\Big(\min_{1\le j\le N: j\ne i}|x_i-x_j|, \lambda\Big). \tag{2.63}$$

Then under the same assumptions as Proposition 2.11, there exists $C>0$ depending only on $\mathsf{d}, \mathsf{s}, p, \phi, \zeta$, such that for every $\eta \le \lambda \le \epsilon$, where ϵ is as in Proposition 2.11, it holds that

$$C\Big(\mathsf{F}_N(X_N,\mu) + \frac{(\mathsf{g}_{\mathrm{Riesz}}(\eta)+C_\rho)}{2N}\mathbf{1}_{\mathsf{s}=0} + C_\zeta\frac{\mathsf{g}_{\mathrm{Riesz}}(\eta)}{2N}\mathbf{1}_{\mathsf{s}>0} + C\|\mu\|_{L^p}\eta^{\frac{\mathsf{d}(p-1)}{p}-\mathsf{s}}\Big) \\ \ge \begin{cases} \frac{1}{N^2}\sum_{i=1}^N \mathsf{g}(\mathsf{r}_i), & \mathsf{s}>0 \\ \frac{1}{N^2}\sum_{i=1}^N \mathsf{g}(\mathsf{r}_i/\eta), & \mathsf{s}=0, \end{cases} \tag{2.64}$$

[6]The value of the sharp constant C is still an open question.

where C_ρ, C_ζ are as in Proposition 2.11.

Proof. Let $\eta \le \lambda$. Discarding the (nonnegative) second and third terms on the left-hand side of (2.60) and unpacking the definition of f_η, we find

$$\begin{aligned}(2.65)\quad \frac{1}{2N^2}\sum_{i=1}^N \sum_{1\le j\le N: |x_i-x_j|\le\eta} (\mathsf{g}(x_i-x_j)-\mathsf{g}_\eta(x_i-x_j)) &\le \frac{1}{2N^2}\sum_{1\le i\ne j\le N} \mathsf{f}_\eta(x_i-x_j) \\ &\le \mathsf{F}_N(X_N,\mu) + \frac{(\mathsf{g}_{\mathrm{Riesz}}(\eta)+C_\rho)}{2N}\mathbf{1}_{\mathsf{s}=0} + C_\zeta \frac{\mathsf{g}_{\mathrm{Riesz}}(\eta)}{2N}\mathbf{1}_{\mathsf{s}>0} + C\|\mu\|_{L^p}\eta^{\frac{\mathsf{d}(p-1)}{p}-\mathsf{s}}.\end{aligned}$$

For each i, either $\mathsf{r}_i = \frac14 \min_{j\ne i}|x_i-x_j|$, in which case there exists $j\ne i$ such that $\mathsf{r}_i = \frac14|x_i-x_j|$, or $\mathsf{r}_i = \frac14\lambda$. In the former case, for such j, we have

$$\begin{aligned}\mathsf{g}(x_i-x_j)-\mathsf{g}_\eta(x_i-x_j) = \mathsf{g}(4\mathsf{r}_i)-\mathsf{g}_\eta(x_i-x_j) &\ge \mathsf{g}(4\mathsf{r}_i)-\mathsf{g}_\eta(0) \\ (2.66)\qquad &\ge (\mathsf{g}(4\mathsf{r}_i)-\mathsf{g}(\eta)) + (\mathsf{g}(\eta)-\mathsf{g}_\eta(0)),\end{aligned}$$

since g_η is decreasing. While in the latter case, for every $j\ne i$,

$$(2.67)\qquad \mathsf{g}(x_i-x_j)-\mathsf{g}_\eta(x_i-x_j) \ge 0 \ge \mathsf{g}(\lambda)-\mathsf{g}(\eta) = \mathsf{g}(4\mathsf{r}_i)-\mathsf{g}(\eta).$$

provided that $\eta\le\lambda$. Thus, in all cases,

$$(2.68)\qquad \sum_{1\le j\le N: |x_i-x_j|\le\eta} (\mathsf{g}(x_i-x_j)-\mathsf{g}_\eta(x_i-x_j)) \ge (\mathsf{g}(4\mathsf{r}_i)-\mathsf{g}(\eta))_+ - |\mathsf{g}(\eta)-\mathsf{g}_\eta(0)|.$$

By Proposition 2.11(1),

$$(2.69)\qquad |\mathsf{g}(\eta)-\mathsf{g}_\eta(0)|$$

Since $\mathsf{f}_\eta\ge 0$, in all cases, we then have using g is decreasing and that g_η is monotone increasing with respect to η that $(\mathsf{g}(x_i-x_j)-\mathsf{g}_\eta(x_i-x_j) \ge (\mathsf{g}(\mathsf{r}_i)=\mathsf{g}(\eta))_+$ for every $j\ne i$, and consequently,

$$(2.70)\qquad \frac{1}{2N^2}\sum_{i=1}^N \sum_{1\le j\le N: |x_i-x_j|\le\eta} (\mathsf{g}(x_i-x_j)-\mathsf{g}_\eta(x_i-x_j)) \ge \frac{1}{2N^2}\sum_{i=1}^N (\mathsf{g}(\mathsf{r}_i)=\mathsf{g}(\eta))_+.$$

□

By following the proof of [RS24c, Proposition 2.5], one may also bound the number of points whose nearest-neighbor distance is at mesoscales. As we do not need such an estimate, we leave the details as an exercise for the reader.

2.4. **Coercivity.** We show in this subsection that the modulated energy associated to an (s,a)-admissible potential is coercive, provided a is sufficiently large, in the sense that it controls a squared negative-order Sobolev norm. In particular, this yields that vanishing of the modulated energy implies weak convergence of the empirical measure to the target measure μ. Such a coercivity estimate is already known in the sub-Coulomb case [**?**] (see also [**?**] for a coercivity estimate in the Coulomb/super-Coulomb case), but its N-dependent additive error is much larger than that shown in the lemma below. Moreover, the estimate presented below is sharper in that the control is $H^{-\frac{\mathsf{d}}{2}-\varepsilon}$, for any $\varepsilon>0$, which is the best one can hope for given the Dirac mass is not in $H^{-\frac{\mathsf{d}}{2}}$.

Proposition 2.14. *Let g be (s,ϕ)-admissible, such that $\hat\phi(\xi)\ge C_\phi\langle\xi\rangle^{-r}$ for some $r>\mathsf{d}$. Suppose $\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}} < p \le \infty$. There exists a constant $C = C(\mathsf{d},\mathsf{s},r,p)$ such that the following holds: for any*

$\mu \in L^1(\mathbb{R}^{\mathsf{d}}) \cap L^p(\mathbb{R}^{\mathsf{d}})$ *with* $\int_{\mathbb{R}^{\mathsf{d}}} d\mu = 1$ *and* $\int_{(\mathbb{R}^{\mathsf{d}})^2} |\mathsf{g}|(x-y)d\mu^{\otimes 2}(x,y) < \infty$ *if* $\mathsf{s} = 0$, $\lambda \leq 1$, *and any pairwise distinct configuration* $X_N \in (\mathbb{R}^{\mathsf{d}})^N$, *it holds that*

$$\left\|\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\right\|_{H^{-r/2}}^2 \leq C\Big(\mathsf{F}_N(X_N,\mu) + \frac{(\mathsf{g}_{\text{Riesz}}(\lambda) + C_\rho)}{2N}\mathbf{1}_{\mathsf{s}=0} + C_\zeta \frac{\mathsf{g}_{\text{Riesz}}(\lambda)}{2N}\mathbf{1}_{\mathsf{s}>0} + C\|\mu\|_{L^p}\lambda^{\frac{\mathsf{d}(p-1)}{p}-\mathsf{s}}\Big), \tag{2.71}$$

where C_ρ, C_ζ *are as in Proposition 2.11. In particular, if* g *is* (s,a)*-admissible for* $\mathsf{a} > \mathsf{d}$*, then* (2.71) *holds with* $r = \mathsf{a}$*; and if* g *is* s*-Riesz, then* (2.71) *holds for any* $r > \mathsf{d}$.

Proof. Recalling the identity (2.13), which holds with $t^{\mathsf{d}-\mathsf{s}}$ replaced by $\zeta(t)$ in the general case, and using that the Fourier transform of $f := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu$ vanishes at the origin, we have that

$$\int_{(\mathbb{R}^{\mathsf{d}})^2} \mathsf{g}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y) = \mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}\int_\eta^\infty \zeta(t)\int_{\mathbb{R}^{\mathsf{d}}} \hat{\phi}(t\xi)|\hat{f}(\xi)|^2 d\xi \frac{dt}{t}, \tag{2.72}$$

where we have also used Fubini-Tonelli to interchange the order of integration. Choose $\hat{\phi}(\xi) = \langle\xi\rangle^{-r}$, i.e. ϕ is a Bessel potential. For $\eta \leq t \leq 1$, we bound $\hat{\phi}(t\xi) \geq \langle\xi\rangle^{-r}$; and for $t \geq 1$, we bound $\hat{\phi}(t\xi) \geq t^{-r}\langle\xi\rangle^{-r}$. Applying these lower bounds to the right-hand side of (2.72),

$$\begin{aligned}\frac{1}{\mathsf{c}_{\phi,\mathsf{d},\mathsf{s}}}\int_{(\mathbb{R}^{\mathsf{d}})^2} \mathsf{g}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y) &\geq \int_\eta^1 \zeta(t)\int_{\mathbb{R}^{\mathsf{d}}} \langle\xi\rangle^{-r}|\hat{f}(\xi)|^2 d\xi\frac{dt}{t} \\ &\quad + \int_1^\infty \zeta(t)t^{-r}\int_{\mathbb{R}^{\mathsf{d}}}\langle\xi\rangle^{-r}|\hat{f}(\xi)|^2 d\xi \frac{dt}{t} \\ &= \frac{1}{C_{\mathsf{d},\mathsf{s},r}}\|f\|_{H^{-\frac{r}{2}}}^2,\end{aligned} \tag{2.73}$$

which requires that $r > \mathsf{d} - \mathsf{s}$ for the second integral in t to converge and $r > \mathsf{d}$ for the integral in ξ to be finite. Bounding the left-hand side of (2.73) from above with the estimate (2.60) from Proposition 2.11 with the choice $\eta = \lambda$, the desired conclusion follows after a little bookkeeping. □

Remark 2.15. If g is (s,ϕ)-admissible where we simply assume that $\hat{\phi} > 0$, then one can still prove that $\mathsf{F}_N(X_N,\mu) \to 0$ implies that $\frac{1}{N}\sum_{i=1}^N \delta_{x_i} \rightharpoonup \mu$ as $N \to \infty$. We leave the proof of this assertion as an exercise for the reader.

3. Kato-Ponce commutator estimates

In this section, we prove some commutator estimates for (fractional) powers of the inhomogeneous Fourier multiplier $\langle\nabla\rangle = (I-\Delta)^{1/2}$. These types of estimates are usually referred to as Kato-Ponce commutator estimates, originating in [KP88]. We refer to [Li19] for a more thorough discussion of the history. Later in Section 4, we will combine these commutator estimates with the potential truncation procedure and the modulated energy bounds of Section 2 to prove Theorem 1.1.

The main results of this section are the following proposition and its corollary.

Proposition 3.1. *Let* $\alpha \geq 0$ *and write* $\alpha = 2m + r$ *for integer* $m \geq 0$ *and* $r \in (0,2]$*. There exists a constant* $C = C(\mathsf{d},\alpha) > 0$ *such that for any* $v, f \in \mathcal{S}(\mathbb{R}^{\mathsf{d}})$,

$$\left|\int_{\mathbb{R}^{\mathsf{d}}} v\cdot\nabla f\langle\nabla\rangle^\alpha f\right| \leq CA_{v,\alpha}\|f\|_{H^{\frac{\alpha}{2}}}^2, \tag{3.1}$$

where

$$A_{v,\alpha} := \|\nabla v\|_{L^\infty} + \sum_{j=0}^{m-1}\Big(\||\nabla|^j \nabla v\|_{L^{\max(\frac{\mathsf{d}}{j},2)}} \mathbf{1}_{j\neq\frac{\mathsf{d}}{2}} + \||\nabla|^j\nabla v\|_{L^{2+}}\mathbf{1}_{j=\frac{\mathsf{d}}{2}} + \||\nabla|^{\frac{r}{2}+j}\nabla v\|_{L^{\max(\frac{2\mathsf{d}}{2j+r},2)}}\mathbf{1}_{j+\frac{r}{2}\neq\frac{\mathsf{d}}{2}} + \||\nabla|^{\frac{r}{2}+j}\nabla v\|_{L^{2+}}\mathbf{1}_{j+\frac{r}{2}=\frac{\mathsf{d}}{2}}\Big). \tag{3.2}$$

Here, the summation is understood as vacuous if $m<1$.

At first glance, the commutator structure in Proposition 3.1 may not be apparent. However, writing $\langle\nabla\rangle^\alpha = \langle\nabla\rangle^{\alpha/2}\langle\nabla\rangle^{\alpha/2}$ and integrating by parts, the left-hand side of (3.1) equals

$$\int_{\mathbb{R}^{\mathsf{d}}} \Big[\langle\nabla\rangle^{\alpha/2}, v\cdot\Big](\nabla f)\langle\nabla\rangle^{\alpha/2} f + \int_{\mathbb{R}^{\mathsf{d}}} v\cdot\nabla\langle\nabla\rangle^{\alpha/2} f\langle\nabla\rangle^{\alpha/2} f. \tag{3.3}$$

The last term is controlled by $\frac{1}{2}\|\nabla v\|_{L^\infty}\|\langle\nabla\rangle^{\alpha/2} f\|_{L^2}^2$ after integrating by parts once more. Hence, if (3.1) holds, then after polarization, it yields the L^2-scale commutator bound $\|\Big[\langle\nabla\rangle^{\alpha/2}, v\cdot\Big]\nabla\|_{H^{\alpha/2}\to L^2} \le CA_{v,\alpha}$.

As a corollary of Proposition 3.1, we obtain the following first-order functional inequality for the truncated potential g_η. We deduce Corollary 3.2 from Proposition 3.1 through the representation (2.7), which exists by our assumption that g is (s, s)-admissible, and averaging over the estimates, noting that $\langle\nabla\rangle^s G_{\mathsf{a}} = \delta_0$.

Corollary 3.2. *Suppose that* g *is* (s,a)*-admissible for some* $\mathsf{a}<\mathsf{d}+2$*. There exists* $C = C(\mathsf{d},\mathsf{s},\mathsf{a})>0$ *such that for every* $\eta>0$*,* $v, f\in\mathcal{S}(\mathbb{R}^{\mathsf{d}})$*,*

$$\Big|\int_{(\mathbb{R}^{\mathsf{d}})^2}(v(x)-v(y))\cdot\nabla\mathsf{g}_\eta(x-y)f(x)f(y)\Big| \le C\Big(\|\nabla v\|_{L^\infty} + \||\nabla|^{\frac{\mathsf{a}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}}\Big)\int_{(\mathbb{R}^{\mathsf{d}})^2}\mathsf{g}_\eta(x-y)f(x)f(y). \tag{3.4}$$

Proof. By the assumption that g is (s,a)-admissible and Fubini-Tonelli, we have that

$$\int_{(\mathbb{R}^{\mathsf{d}})^2}(v(x)-v(y))\cdot\nabla\mathsf{g}_\eta(x-y)f(x)f(y) = \int_\eta^\infty\frac{\zeta(t)}{t}\int_{(\mathbb{R}^{\mathsf{d}})^2}(v(x)-v(y))\cdot\nabla G_{\mathsf{a}}(\frac{x-y}{t})f(x)f(y)\frac{dt}{t}. \tag{3.5}$$

Making the change of variable $tz=x$ and $ty=y$ and letting $v_t(x) := v(tx)$ and $f_t(x) := f(tx)$, we find that

$$\int_{(\mathbb{R}^{\mathsf{d}})^2}(v(x)-v(y))\cdot\nabla G_{\mathsf{a}}(\frac{x-y}{t})f(x)f(y) = t^{2\mathsf{d}}\int_{(\mathbb{R}^{\mathsf{d}})^2}(v_t(z)-v_t(w))\cdot\nabla G_{\mathsf{a}}(w-z)f_t(z)f_t(w)$$

$$= 2t^{2\mathsf{d}}\int_{\mathbb{R}^{\mathsf{d}}} v_t\cdot\nabla(G_{\mathsf{a}} * f_t)f_t, \tag{3.6}$$

where the final line follows from desymmetrizing. Letting $h_t := G_{\mathsf{a}} * f_t$, noting that $\langle\nabla\rangle^s h_t = f_t$, and applying Proposition 3.1 with f replaced by h_t, we obtain

$$\Big|\int_{\mathbb{R}^{\mathsf{d}}} v_t\cdot\nabla h_t f_t\Big| \le CA_{v_t,s}\|h_t\|_{H^{\frac{s}{2}}}^2. \tag{3.7}$$

Unpacking the definition of h_t and undoing the change of variables made above,

$$\|h_t\|_{H^{\frac{s}{2}}}^2 = \int_{(\mathbb{R}^{\mathsf{d}})^2} G_{\mathsf{a}}(x-y)f_t(x)f_t(y) = t^{-2\mathsf{d}}\int_{(\mathbb{R}^{\mathsf{d}})^2} G_{\mathsf{a}}(\frac{x-y}{t})f(x)f(y). \tag{3.8}$$

Write $s = 2m + r$ for integer $m \geq 0$ and $r \in (0, 2]$. Unpacking the definition (3.2) of $A_{v_t,s}$,

$$\begin{aligned} A_{v_t,s} &= \|\nabla v_t\|_{L^\infty} + \sum_{0\leq j\leq m-1} \Big(\||\nabla|^j \nabla v_t\|_{L^{\max(\frac{\mathsf{d}}{j},2)}} \mathbf{1}_{j\neq\frac{\mathsf{d}}{2}} + \||\nabla|^{\frac{r}{2}+j}\nabla v_t\|_{L^{\max(\frac{2\mathsf{d}}{2j+r},2)}} \mathbf{1}_{j+\frac{r}{2}\neq\frac{\mathsf{d}}{2}} \\ &\quad + \||\nabla|^j\nabla v_t\|_{L^{2+}} \mathbf{1}_{j=\frac{\mathsf{d}}{2}} + \||\nabla|^{\frac{r}{2}+j}\nabla v_t\|_{L^{2+}} \mathbf{1}_{j+\frac{r}{2}=\frac{\mathsf{d}}{2}} \Big) \\ &= \sum_{0\leq j\leq m-1} \Big(t^{j+1-\frac{\mathsf{d}}{\max(\mathsf{d}/j,2)}} \||\nabla|^j \nabla v\|_{L^{\max(\frac{\mathsf{d}}{j},2)}} \mathbf{1}_{j\neq\frac{\mathsf{d}}{2}} + \||\nabla|^{\frac{r}{2}+j}\nabla v_t\|_{L^{\max(\frac{2\mathsf{d}}{2j+r},2)}} \mathbf{1}_{j+\frac{r}{2}\neq\frac{\mathsf{d}}{2}} \\ &\quad + \||\nabla|^j\nabla v_t\|_{L^{2+}} \mathbf{1}_{j=\frac{\mathsf{d}}{2}} + \||\nabla|^{\frac{r}{2}+j}\nabla v_t\|_{L^{2+}} \mathbf{1}_{j+\frac{r}{2}=\frac{\mathsf{d}}{2}} \Big) \end{aligned} \tag{3.9}$$

The preceding expression does not scale properly in t (i.e. scales like t) if $m - 1 + \frac{r}{2} \geq \frac{\mathsf{d}}{2}$. This motivates our assumption that $s < \mathsf{d} + 2$, which is equivalent to $m - 1 + \frac{r}{2} < \frac{\mathsf{d}}{2}$. Hence,

$$A_{v_t,s} \leq t\Big(\|\nabla v\|_{L^\infty} + \sum_{j=0}^{m-1} \||\nabla|^j\nabla v\|_{L^{\frac{\mathsf{d}}{j}}} + \||\nabla|^{\frac{r}{2}+j}\nabla v\|_{L^{\frac{2\mathsf{d}}{2j+r}}} \Big) \leq Ct\Big(\|\nabla v\|_{L^\infty} + \||\nabla|^{m+\frac{r}{2}} v\|_{L^{\frac{2\mathsf{d}}{2(m-1)+r}}} \Big), \tag{3.10}$$

where the final inequality is by Sobolev embedding. From the relations (3.7), (3.8), (3.10), we obtain

$$\left| \int_{\mathbb{R}^\mathsf{d}} v_t \cdot \nabla h_t f_t \right| \leq C t^{-2\mathsf{d}-1} \Big(\|\nabla v\|_{L^\infty} + \||\nabla|^{m+\frac{r}{2}} v\|_{L^{\frac{2\mathsf{d}}{2(m-1)+r}}} \Big) \int_{(\mathbb{R}^\mathsf{d})^2} G_\mathsf{a}(\frac{x-y}{t}) f(x) f(y). \tag{3.11}$$

Note that $m + \frac{r}{2} = \frac{s}{2}$ and $\frac{2\mathsf{d}}{2(m-1)+r} = \frac{2\mathsf{d}}{s-2}$.

Combining the relations (3.5), (3.6), (3.11), we arrive at

$$\begin{aligned} &\left| \int_{(\mathbb{R}^\mathsf{d})^2} (v(x) - v(y)) \cdot \nabla \mathsf{g}_\eta(x-y) f(x) f(y) \right| \\ &\leq \int_\eta^\infty \frac{\zeta(t)}{t} t^{2\mathsf{d}} \int_{(\mathbb{R}^\mathsf{d})^2} t^{-2\mathsf{d}+1} \Big(\|\nabla v\|_{L^\infty} + \||\nabla|^{\frac{s}{2}} v\|_{L^{\frac{2\mathsf{d}}{s-2}}} \Big) G_\mathsf{a}(\frac{x-y}{t}) f(x) f(y) \frac{dt}{t} \\ &= \Big(\|\nabla v\|_{L^\infty} + \||\nabla|^{\frac{s}{2}} v\|_{L^{\frac{2\mathsf{d}}{s-2}}} \Big) \int_{(\mathbb{R}^\mathsf{d})^2} \mathsf{g}_\eta(x-y) f(x) f(y), \end{aligned} \tag{3.12}$$

which is exactly the desired conclusion. ☐

Remark 3.3. By standard density arguments (e.g. see ?), Proposition 3.1 extends to hold for v, f such that $A_{\nabla v,m,r} < \infty$ and $f \in H^{\frac{\alpha}{2}}$. Similarly, Corollary 3.2 extends to hold for v such that $\|\nabla v\|_{L^\infty} + \||\nabla|^{\frac{\mathsf{a}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}} < \infty$ and any distribution f such that the pairing $\langle \mathsf{g}_\eta(x-y), f^{\otimes 2}\rangle < \infty$.

To prove Proposition 3.1, we first need to establish some preliminary lemmas. The first lemma is a technical result, which may be viewed as an inhomogeneous fractional Leibniz rule.

Lemma 3.4. *For any $0 < r \leq 2$ and multi-index $\vec{\alpha} \in \mathbb{N}_0^\mathsf{d}$, there exists $C = C(\mathsf{d}, r, |\vec{\alpha}|) > 0$ such that for $f, g \in \mathcal{S}(\mathbb{R}^\mathsf{d})$,*

$$\|\langle\nabla\rangle^{r/2} \partial_{\vec{\alpha}}(fg)\|_{L^2} \leq C \|g\|_{H^{|\vec{\alpha}|+\frac{r}{2}}} \tilde{A}_{f,|\vec{\alpha}|,r}, \tag{3.13}$$

where $\tilde{A}_{f,|\vec{\alpha}|,r}$ is defined by

$$\begin{aligned} \tilde{A}_{f,|\vec{\alpha}|,r} &:= \sum_{0\leq j\leq|\vec{\alpha}|} \Big(\||\nabla|^j f\|_{L^{\max(\frac{\mathsf{d}}{j},2)}} \mathbf{1}_{j\neq\frac{\mathsf{d}}{2}} + \||\nabla|^j f\|_{L^{2+}} \mathbf{1}_{j=\frac{\mathsf{d}}{2}} \Big) \\ &\quad + \sum_{0\leq j\leq|\vec{\alpha}|} \Big(\||\nabla|^{\frac{r}{2}+j} f\|_{L^{\max(\frac{2\mathsf{d}}{2j+r},2)}} \mathbf{1}_{j+\frac{r}{2}\neq\frac{\mathsf{d}}{2}} + \||\nabla|^{\frac{r}{2}+j} f\|_{L^{2+}} \mathbf{1}_{j+\frac{r}{2}=\frac{\mathsf{d}}{2}} \Big). \end{aligned} \tag{3.14}$$

In particular, if $|\vec{\alpha}| + \frac{r}{2} < \frac{\mathsf{d}}{2}$*, then* $\tilde{A}_{f,|\vec{\alpha}|,m} \leq C(\|f\|_{L^\infty} + \||\nabla|^{|\vec{\alpha}|+\frac{r}{2}} f\|_{L^{\frac{\mathsf{d}}{|\vec{\alpha}|+\frac{r}{2}}}})$.

Proof. Set $m := |\vec{\alpha}|$. Applying the Leibniz rule to $\partial_{\vec{\alpha}}$ and using the triangle inequality, we find that

$$\|\langle\nabla\rangle^{r/2}\partial_{\vec{\alpha}}(fg)\|_{L^2} \leq \sum_{\vec{\beta}\leq\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}} \|\langle\nabla\rangle^{r/2}(\partial_{\vec{\beta}} f \partial_{\vec{\alpha}-\vec{\beta}} g)\|_{L^2} \tag{3.15}$$

$$\leq \sum_{\vec{\beta}\leq\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}} \|\langle\nabla\rangle^{r/2}(\partial_{\vec{\beta}} f \partial_{\vec{\alpha}-\vec{\beta}} g) - \partial_{\vec{\beta}} f \langle\nabla\rangle^{r/2}\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^2} + \|\partial_{\vec{\beta}} f \langle\nabla\rangle^{r/2}\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^2}. \tag{3.16}$$

Appealing to the Kato-Ponce estimate [Li19, Theorem 1.9], we have that

$$\|\langle\nabla\rangle^{r/2}(\partial_{\vec{\beta}} f \partial_{\vec{\alpha}-\vec{\beta}} g) - \partial_{\vec{\beta}} f \langle\nabla\rangle^{r/2}\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^2} \leq \|\langle\nabla\rangle^{\frac{r}{2}-1}\nabla\partial_{\vec{\beta}} f\|_{L^{p_1}} \|\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^{p_2}} \tag{3.17}$$

for any $1 < p_1, p_2 \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. Since $\frac{r}{2} - 1 \leq 0$ and the Fourier multipliers $\frac{\nabla}{\langle\nabla\rangle}, \frac{\nabla}{|\nabla|}$ are bounded on L^p for any $1 < p < \infty$ by the Hörmander-Mikhlin theorem, we see that

$$\|\langle\nabla\rangle^{\frac{r}{2}-1}\nabla\partial_{\vec{\beta}} f\|_{L^{p_1}} \leq C\||\nabla|^{\frac{r}{2}+|\vec{\beta}|} f\|_{L^{p_1}}. \tag{3.18}$$

We choose p_1, p_2 according to

$$(p_1, p_2) = \begin{cases} (\frac{2\mathsf{d}}{2|\vec{\beta}|+r}, \frac{2\mathsf{d}}{\mathsf{d}-2|\vec{\beta}|-r}), & |\vec{\beta}| + \frac{r}{2} < \frac{\mathsf{d}}{2} \\ (2+\epsilon, \frac{2(2+\epsilon)}{\epsilon}), & |\vec{\beta}| + \frac{r}{2} = \frac{\mathsf{d}}{2} \\ (2, \infty), & |\vec{\beta}| + \frac{r}{2} > \frac{\mathsf{d}}{2}, \end{cases} \tag{3.19}$$

where $\epsilon > 0$ is arbitrary. In which case, it follows follows from Sobolev embedding applied to $\|\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^{p_2}}$ that

$$\|\langle\nabla\rangle^{r/2}(\partial_{\vec{\beta}} f \partial_{\vec{\alpha}-\vec{\beta}} g) - \partial_{\vec{\beta}} f \langle\nabla\rangle^{r/2}\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^2} \leq C\|g\|_{H^{m+\frac{r}{2}}} \begin{cases} \||\nabla|^{\frac{r}{2}+|\vec{\beta}|} f\|_{L^{\frac{2\mathsf{d}}{2|\vec{\beta}|+r}}}, & |\vec{\beta}| + \frac{r}{2} < \frac{\mathsf{d}}{2} \\ \||\nabla|^{\frac{r}{2}+|\vec{\beta}|} f\|_{L^{2+\epsilon}}, & |\vec{\beta}| + \frac{r}{2} = \frac{\mathsf{d}}{2} \\ \||\nabla|^{\frac{r}{2}+|\vec{\beta}|} f\|_{L^2}, & |\vec{\beta}| + \frac{r}{2} > \frac{\mathsf{d}}{2}. \end{cases} \tag{3.20}$$

On the other hand, Hlder's inequality implies that for any $1 \leq p_3, p_4 \leq \infty$ such that $\frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{2}$,

$$\|\partial_{\vec{\beta}} f \langle\nabla\rangle^{r/2}\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^2} \leq \|\partial_{\vec{\beta}} f\|_{L^{p_3}} \|\langle\nabla\rangle^{r/2}\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^{p_4}}. \tag{3.21}$$

We choose (p_3, p_4) according to

$$(p_3, p_4) = \begin{cases} (\frac{\mathsf{d}}{|\vec{\beta}|}, \frac{2\mathsf{d}}{\mathsf{d}-2|\vec{\beta}|}), & |\vec{\beta}| < \frac{\mathsf{d}}{2} \\ (2+\epsilon', \frac{2(2+\epsilon')}{\epsilon'}), & |\vec{\beta}| = \frac{\mathsf{d}}{2} \\ (2, \infty), & |\vec{\beta}| > \frac{\mathsf{d}}{2}, \end{cases} \tag{3.22}$$

where $\epsilon' > 0$ is again arbitrary. Applying Sobolev embedding again to $\|\langle\nabla\rangle^{r/2}\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^{p_4}}$, it follows that

$$\|\partial_{\vec{\beta}} f \langle\nabla\rangle^{r/2}\partial_{\vec{\alpha}-\vec{\beta}} g\|_{L^2} \leq C\|g\|_{H^{m+\frac{r}{2}}} \begin{cases} \||\nabla|^{|\vec{\beta}|} f\|_{L^{\frac{\mathsf{d}}{|\vec{\beta}|}}}, & |\vec{\beta}| < \frac{\mathsf{d}}{2} \\ \||\nabla|^{|\vec{\beta}|} f\|_{L^{2+\epsilon'}}, & |\vec{\beta}| = \frac{\mathsf{d}}{2} \\ \||\nabla|^{|\vec{\beta}|} f\|_{L^2}, & |\vec{\beta}| > \frac{\mathsf{d}}{2}. \end{cases} \tag{3.23}$$

If $m+\frac{r}{2}<\frac{\mathsf{d}}{2}$, then by Gagliardo-Nirenberg interpolation [?, Theorem 2.44],

$$\||\nabla|^{|\vec{\beta}|}f\|_{L^{\frac{\mathsf{d}}{|\vec{\beta}|}}} \le C\|f\|_{L^\infty}^{1-\frac{|\vec{\beta}|}{m+\frac{r}{2}}}\||\nabla|^{m+\frac{r}{2}}f\|_{L^{\frac{2\mathsf{d}}{2m+r}}}^{\frac{|\vec{\beta}|}{m+\frac{r}{2}}}, \tag{3.24}$$

$$\||\nabla|^{|\vec{\beta}|+\frac{r}{2}}f\|_{L^{\frac{2\mathsf{d}}{2|\vec{\beta}|+r}}} \le C\|f\|_{L^\infty}^{1-\frac{2|\vec{\beta}|+r}{2m+r}}\||\nabla|^{m+\frac{r}{2}}f\|_{L^{\frac{2\mathsf{d}}{2m+r}}}^{\frac{2|\vec{\beta}|+r}{2m+r}}. \tag{3.25}$$

The desired conclusion follows from Young's inequality and a little bookkeeping.

□

The second result we need is a representation of fractional powers $\langle\nabla\rangle^{s/2}$ as the Dirichlet-to-Neumann map of an extension problem to $\mathbb{R}^{\mathsf{d}+1}$ for a (degenerate elliptic) second-order partial differential operator, which may be viewed as an inhomogeneous analogue of the celebrated Caffarelli-Silvestre extension for the fractional Laplacian [CS07]. Such a dimension extension is a special case of a more general theory established in [ST10] (see also [MN24] for a substantial generalization to operators on Hilbert spaces).

Lemma 3.5. *Let $s\in(0,2)$ and $f\in\mathcal{S}(\mathbb{R}^{\mathsf{d}})$. A solution of the Dirichlet problem*

$$\begin{cases}(\Delta_x - I)F + \frac{(1-s)}{z}\partial_z F + \partial_z^2 F = 0, & \text{in } \mathbb{R}^{\mathsf{d}}\times(0,\infty)\\ F(x,z)=f(x), & \text{on } \mathbb{R}^{\mathsf{d}}\times\{0\}\end{cases} \tag{3.26}$$

is given by the Poisson formula

$$F(x,z) = \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty e^{-t}\Big(e^{\frac{z^2}{4t}(\Delta-I)}f\Big)(x)\frac{dt}{t^{1-\frac{s}{2}}} \tag{3.27}$$

and

$$\lim_{z\to0^+}\frac{F(x,z)-F(x,0)}{z^s} = \frac{\Gamma(-\frac{s}{2})}{2^s\Gamma(\frac{s}{2})}\langle\nabla\rangle^s f(x) = \frac{1}{s}\lim_{z\to0^+} z^{1-s}\partial_z F(x,z), \tag{3.28}$$

where the convergence holds both pointwise and in $L^2(\mathbb{R}^{\mathsf{d}})$. Consequently, extending F to $\mathbb{R}^{\mathsf{d}+1}$ by even symmetry, we have

$$|z|^{1-s}F - \operatorname{div}(|z|^{1-s}\nabla F) = \mathsf{c}_{\mathsf{d},s}\langle\nabla\rangle^s f\delta_{\mathbb{R}^{\mathsf{d}}\times\{0\}} \tag{3.29}$$

in the sense of distributions in $\mathbb{R}^{\mathsf{d}+1}$ and

$$\int_{\mathbb{R}^{\mathsf{d}+1}}|z|^{1-s}(F+|\nabla F|^2) = \|f\|_{H^{\frac{s}{2}}}^2. \tag{3.30}$$

Proof. The assertions (3.27)-(3.28) are the content of [ST10, Theorem 1.1]. To see the assertion (3.29), let F be a solution of (3.26) and let Φ be a compactly supported test function in $\mathbb{R}^{\mathsf{d}+1}$. By dominated convergence,

$$\begin{aligned}&\int_{\mathbb{R}^{\mathsf{d}+1}}\Big(|z|^{1-s}\Phi - \operatorname{div}(|z|^{1-s}\nabla\Phi)\Big)F\\ &= \lim_{\epsilon\to0^+}\left(\int_{\{|z|\ge\epsilon\}\cap B_{1/\epsilon}}|z|^{1-s}\Phi F - \int_{\{|z|\ge\epsilon\}\cap B_{1/\epsilon}}\operatorname{div}(|z|^{1-s}\nabla\Phi)F\right),\end{aligned} \tag{3.31}$$

where the balls are in $\mathbb{R}^{\mathsf{d}+1}$. Integrating by parts twice, we see that

$$\begin{aligned}-\int_{\{|z|\ge\epsilon\}\cap B_{1/\epsilon}}\operatorname{div}(|z|^{1-s}\nabla\Phi)F &= -\int_{\{|z|=\epsilon\}}|z|^{1-s}\partial_\nu\Phi F - \int_{\{|z|\ge\epsilon\}\cap\partial B_{1/\epsilon}}|z|^{1-s}\partial_\nu\Phi F\\ &+\int_{\{|z|=\epsilon\}}|z|^{1-s}\Phi\partial_\nu F - \int_{\{|z|\ge\epsilon\}\cap\partial B_{1/\epsilon}}|z|^{1-s}\Phi\partial_\nu F - \int_{\{|z|\ge\epsilon\}\cap B_{1/\epsilon}}\Phi\operatorname{div}(|z|^{1-s}\nabla F).\end{aligned} \tag{3.32}$$

It is immediate from (3.26) that

$$0 = \int_{\{|z|\geq\epsilon\}\cap B_{1/\epsilon}} z^{1-s}\Phi F - \int_{\{|z|\geq\epsilon\}\cap B_{1/\epsilon}} \Phi\,\mathrm{div}(|z|^{1-s}\nabla F). \tag{3.33}$$

The boundary contributions from $\{|z|\geq\epsilon\}\cap\partial B_{1/\epsilon}$ vanish, provided ϵ is small enough, since Φ has compact support. Now using that F is even in the z coordinate,

$$\begin{aligned}\left|\int_{\{|z|=\epsilon\}} |z|^{1-s}\partial_\nu\Phi F\right| &= \epsilon^{1-s}\left|\int_{\mathbb{R}^{\mathsf{d}}}\Big(\partial_z\Phi(\cdot,\epsilon)-\partial_z\Phi(\cdot,-\epsilon)\Big)F(\cdot,\epsilon)\right| \\ &\leq 2\|\sup_{|z|\leq\epsilon}\partial_z^2\Phi(\cdot,z)\|_{L^1}\|F\|_{L^\infty}\epsilon^{2-s},\end{aligned} \tag{3.34}$$

where we use the mean-value theorem to obtain the last line. That $\|F\|_{L^\infty}<\infty$ follows from Lemma 3.6 below. Since $s<2$, the preceding expression vanishes as $\epsilon\to0^+$. For the remaining boundary term, observe that

$$\int_{\{|z|=\epsilon\}}|z|^{1-s}\Phi\partial_\nu F = -\int_{\mathbb{R}^{\mathsf{d}}}\epsilon^{1-s}\Big(\Phi(\cdot,\epsilon)\partial_z F(\cdot,\epsilon)-\Phi(\cdot,-\epsilon)\partial_z F(\cdot,-\epsilon)\Big). \tag{3.35}$$

Using that $\frac{1}{s}z^{1-s}\partial_z F(\cdot,z)$ converges to $\frac{\Gamma(-\frac{s}{2})}{2^s\Gamma(\frac{s}{2})}\langle\nabla\rangle^s f$ in $L^2(\mathbb{R}^{\mathsf{d}})$ as $z\to0^+$, it follows that the preceding right-hand side converges to

$$2^{1-s}s\frac{\Gamma(-s/2)}{\Gamma(s/2)}\int_{\mathbb{R}^{\mathsf{d}}}\Phi(\cdot,0)\langle\nabla\rangle^s f \tag{3.36}$$

as $\epsilon\to0^+$. After a little bookkeeping, we conclude that

$$\int_{\mathbb{R}^{\mathsf{d}+1}}\Big(|z|^{1-s}\Phi-\mathrm{div}(|z|^{1-s}\nabla\Phi)\Big)F = 2^{1-s}s\frac{\Gamma(-s/2)}{\Gamma(s/2)}\int_{\mathbb{R}^{\mathsf{d}}}\Phi(\cdot,0)\langle\nabla\rangle^s f. \tag{3.37}$$

As $\Phi\in C_c^\infty(\mathbb{R}^{\mathsf{d}+1})$ was arbitrary, this establishes the assertion (3.29).

To see the last assertion (3.30), we note that (3.29) implies that

(3.38)
$$\|f\|_{H^{s/2}}^2 = \int_{\mathbb{R}^{\mathsf{d}}} f\langle\nabla\rangle^s f = \frac{1}{\mathsf{c}_{\mathsf{d},s}}\int_{\mathbb{R}^{\mathsf{d}+1}}F\Big(|z|^{1-s}F-\mathrm{div}(|z|^{1-s}\nabla F)\Big) = \frac{1}{\mathsf{c}_{\mathsf{d},s}}\int_{\mathbb{R}^{\mathsf{d}+1}}|z|^{1-s}\Big(|F|^2+|\nabla F|^2\Big),$$

where the last equality follows from an integration by parts, which is justified by Lemma 3.6 below. The proof of the lemma is now complete. □

The following lemma provides regularity and decay estimates for the Poisson extension F in terms of the boundary data f. In particular, if $f\in\mathcal{S}(\mathbb{R}^{\mathsf{d}})$, which we may always reduce to through density, there are no issues of regularity/decay in the computations above. The lemma is more general than needed, but some of the estimates do not seem to be written down anywhere in the literature and are perhaps of future use.

Lemma 3.6.

Proof. Given $f\in L^p(\mathbb{R}^{\mathsf{d}})$, for $1\leq p\leq\infty$, it follows immediately from Young's inequality that $\sup_{z>0}\|F(\cdot,z)\|_{L^p}\leq C\|f\|_{L^p}$. Moreover, since ∇_x commutes with $e^{\frac{z^2}{4t}\Delta}$, the same is true with f replaced by $\nabla_x^{\otimes k}f$.

For decay in z, we decompose into cases $t\leq z^2$ and $t>z^2$. In the former, we pointwise bound $|e^{\frac{z^2}{4t}\Delta}f(x)|\leq C\frac{t^{\mathsf{d}/2}}{z^{\mathsf{d}}}\|f\|_{L^1}$. In the latter, we crudely bound $|e^{\frac{z^2}{4t}\Delta}f(x)|\leq\|f\|_{L^\infty}$. Decomposing

$\int_0^\infty = \int_0^{z^2} + \int_{z^2}^\infty$, it then follows that if $\frac{\mathsf{d}+s}{2} - 1 \geq 0$,

$$
\begin{aligned}
|F(x,z)| &\leq \frac{C}{\Gamma(\frac{s}{2})}\int_0^{z^2} e^{-t}\frac{t^{\mathsf{d}/2}\|f\|_{L^1}}{z^{\mathsf{d}}}\frac{dt}{t^{1-\frac{s}{2}}} + \frac{C}{\Gamma(\frac{s}{2})}\int_{z^2}^\infty \frac{e^{-\epsilon z^2}}{z^{2-s}}\|f\|_{L^\infty}e^{-(1-\epsilon)t}dt \\
&\leq C_{\mathsf{d},s,\epsilon} z^{-\mathsf{d}}\|f\|_{L^1}\int_0^\infty e^{-(1-\epsilon)t}dt + C_{s,\epsilon}z^{-(2-s)}e^{-\epsilon z^2}\|f\|_{L^\infty}\int_0^\infty e^{-(1-\epsilon)t}dt,
\end{aligned} \tag{3.39}
$$

for any $\epsilon \in (0,1)$, where we have used that $s \in (0,2)$ and $|u|^m e^{-|u|} \leq C_{m,\epsilon}e^{-(1-\epsilon)|u|}$ for any $m \geq 0, \epsilon \in (0,1)$. If $\mathsf{d} = 1$, then the previous argument does not quite work, given $\frac{\mathsf{d}}{2} + \frac{s}{2} - 1 = \frac{s-1}{2}$, which may be negative. For $\delta \in [0,1)$, decompose $\int_0^\infty = \int_0^{z^{2(1-\delta)}} + \int_{z^{2(1-\delta)}}^\infty$ to instead obtain

$$
\begin{aligned}
|F(x,z)| &\leq \frac{C}{\Gamma(\frac{s}{2})}\int_0^{z^{2(1-\delta)}} e^{-t}\frac{t^{\mathsf{d}/2}\|f\|_{L^1}}{z^{\mathsf{d}}}\frac{dt}{t^{1-\frac{s}{2}}} + \frac{C}{\Gamma(\frac{s}{2})}\int_{z^{2(1-\delta)}}^\infty \frac{e^{-\epsilon z^{2(1-\delta)}}}{z^{(2-s)(1-\delta)}}\|f\|_{L^\infty}e^{-(1-\epsilon)t}dt \\
&\leq C_{\mathsf{d},s,\delta} z^{(1-\delta)(\mathsf{d}+\mathsf{s})-\mathsf{d}}\|f\|_{L^1} + C_{s,\delta,\epsilon}\frac{e^{-\epsilon z^{2(1-\delta)}}}{z^{(2-s)(1-\delta)}}\|f\|_{L^\infty}
\end{aligned} \tag{3.40}
$$

As we may repeat the logic with f replaced by $\nabla_x^{\otimes k} f$, we thus obtain that

$$
\begin{aligned}
|\nabla_x^{\otimes k}F(x,z)| &\leq \Big(C_{\mathsf{d},s,\epsilon}z^{-\mathsf{d}}\|\nabla_x^{\otimes k}f\|_{L^1} + C_{s,\epsilon}z^{-(2-s)}e^{-\epsilon z^2}\|\nabla_x^{\otimes k}f\|_{L^\infty}\Big)\mathbf{1}_{\frac{\mathsf{d}+s}{2}\geq 1} \\
&\qquad + \Big(C_{\mathsf{d},s,\delta}z^{(1-\delta)(\mathsf{d}+\mathsf{s})-\mathsf{d}}\|f\|_{L^1} + C_{s,\delta,\epsilon}\frac{e^{-\epsilon z^{2(1-\delta)}}}{z^{(2-s)(1-\delta)}}\|f\|_{L^\infty}\Big)\mathbf{1}_{\frac{\mathsf{d}+s}{2}<1}.
\end{aligned} \tag{3.41}
$$

For decay in x, we write

$$
e^{\frac{z^2}{4t}\Delta}f(x) = (z^2/4t)^{-\frac{\mathsf{d}}{2}}\int_{\mathbb{R}^{\mathsf{d}}} e^{-\frac{4t}{z^2}|x-y|^2}f(y). \tag{3.42}
$$

Split $\int_{\mathbb{R}^{\mathsf{d}}} = \int_{|y|\leq\frac{1}{2}|x|} + \int_{|y|>\frac{1}{2}|x|}$. For the region $|y| > \frac{1}{2}|x|$, then trivially,

$$
\begin{aligned}
(z^2/4t)^{-\frac{\mathsf{d}}{2}}\int_{|y|>\frac{1}{2}|x|} e^{-\frac{4t}{z^2}|x-y|^2}|f(y)| &\leq \|\langle\cdot\rangle^m f\|_{L^\infty}\langle x\rangle^{-m}C(z^2/4t)^{-\frac{\mathsf{d}}{2}}\int_{|y|>\frac{1}{2}|x|} e^{-\frac{4t}{z^2}|x-y|^2} \\
&\leq C\|\langle\cdot\rangle^m f\|_{L^\infty}\langle x\rangle^{-m}.
\end{aligned} \tag{3.43}
$$

Hence,

$$
\frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty e^{-t-\frac{z^2}{4t}}(z^2/4t)^{-\frac{\mathsf{d}}{2}}\int_{|y|>\frac{1}{2}|x|} e^{-\frac{4t}{z^2}|x-y|^2}|f(y)|dy\frac{dt}{t^{1-\frac{s}{2}}} \leq C_s\|\langle\cdot\rangle^m f\|_{L^\infty}\langle x\rangle^{-m} \tag{3.44}
$$

For the contribution of the $\{|y| \leq \frac{1}{2}|x|\}$, first note that $e^{-\frac{2t}{z^2}|x-y|^2} \leq e^{-\frac{t}{2z^2}|x|^2}$. Now introduce $\delta \in (0,1)$ and consider the cases $\frac{z^2}{t} \leq |x|^{2(1-\delta)}$ and $\frac{z^2}{t} > |x|^{2(1-\delta)}$. In the former, we use that $e^{-\frac{t}{2z^2}|x|^2} \leq e^{-|x|^{2\delta}}$; and in the latter, we use that $e^{-\frac{z^2}{4t}} \leq e^{-\frac{1}{4}|x|^{2(1-\delta)}}$. All together, we find that

$$
\begin{aligned}
&\frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty e^{-t-\frac{z^2}{4t}}(z^2/4t)^{-\frac{\mathsf{d}}{2}}\int_{|y|\leq\frac{1}{2}|x|} e^{-\frac{4t}{z^2}|x-y|^2}|f(y)|dy\frac{dt}{t^{1-\frac{s}{2}}} \\
&\leq C_s\|f\|_{L^\infty}\int_0^\infty e^{-\frac{t}{2z^2}|x|^2}e^{-t-\frac{z^2}{4t}}\frac{dt}{t^{1-\frac{s}{2}}} \\
&\leq C_s\|f\|_{L^\infty}\left(\int_0^{\frac{z^2}{|x|^{2(1-\delta)}}} e^{-\frac{1}{4}|x|^{2(1-\delta)}}e^{-\frac{t}{2z^2}|x|^2}e^{-t}\frac{dt}{t^{1-\frac{s}{2}}} + \int_{\frac{z^2}{|x|^{2(1-\delta)}}}^\infty e^{-|x|^{2\delta}}e^{-t-\frac{z^2}{4t}}\frac{dt}{t^{1-\frac{s}{2}}}\right) \\
&\leq C_s'\|f\|_{L^\infty}\Big(e^{-\frac{1}{4}|x|^{2(1-\delta)}} + e^{-|x|^{2\delta}}\Big).
\end{aligned} \tag{3.45}
$$

The two terms on the final line may be balanced by choosing $\delta = \frac{1}{2}$.

We now turn to $\partial_z F$. For $z > 0$, making the change of variable $t/z^2 \mapsto t$, we rewrite (3.27) as

$$F(x,z) = \frac{z^s}{\Gamma(\frac{s}{2})}\int_0^\infty e^{-z^2 t}\Big(e^{\frac{1}{4t}(\Delta - I)}f\Big)(x)\frac{dt}{t^{1-\frac{s}{2}}}. \tag{3.46}$$

Hence,

$$\begin{aligned}\partial_z F(x,z) &= \frac{sz^{s-1}}{\Gamma(\frac{s}{2})}\int_0^\infty e^{-z^2 t}\Big(e^{\frac{1}{4t}(\Delta - I)}f\Big)(x)\frac{dt}{t^{1-\frac{s}{2}}} \\ &\quad - \frac{2z^{s+1}}{\Gamma(\frac{s}{2})}\int_0^\infty te^{-z^2 t}\Big(e^{\frac{1}{4t}(\Delta - I)}f\Big)(x)\frac{dt}{t^{1-\frac{s}{2}}} \\ &= sz^{-1}F(x,z)\end{aligned} \tag{3.47}$$

Note that since for any $\epsilon > 0$, $ue^{-u} \le C_\epsilon e^{-(1-\epsilon)u}$, it follows that

$$\begin{aligned}\frac{2z^{s+1}}{\Gamma(\frac{s}{2})}\int_0^\infty te^{-z^2 t}\left|\Big(e^{\frac{1}{4t}(\Delta - I)}f\Big)\right|(x)\frac{dt}{t^{1-\frac{s}{2}}} &\le 2C_\epsilon\frac{z^{s-1}}{\Gamma(\frac{s}{2})}\int_0^\infty e^{-(1-\epsilon)z^2 t}\Big(e^{\frac{1}{4t}(\Delta - I)}|f|\Big)(x)\frac{dt}{t^{1-\frac{s}{2}}} \\ &= C'_\epsilon z^{-1}\tilde{F}(x,\sqrt{1-\epsilon}z),\end{aligned} \tag{3.48}$$

where $\tilde{F}$ is (3.27) with f replaced by $|f|$. As $\sup_{z'>0}|\tilde{F}(x,z')| \le \|f\|_\infty$, it follows from triangle inequality that. Similarly, replacing f by $\nabla_x^{\otimes k}f$ in the starting point, we in fact have shown that $\sup_{z>0} z|\nabla_x^{\otimes k}\partial_z F(x,z)| < \|f\|_{L^\infty}$. □

We now turn to the proof of Proposition 3.1.

Proof of Proposition 3.1. Write $\alpha = 2m + r$ for integer $m \ge 0$ and $r \in (0,2]$. We will prove by induction on m that if Proposition 3.1 is true for all $\alpha = 2m + r$, for $r \in (0,2]$, then it is true for all $\alpha = 2(m+1) + r$, for $r \in (0,2]$.

We begin with the base case $m = 0$. Given the datum f, let F denote the extension to $\mathbb{R}^{\mathsf{d}+1}$ given by Lemma 3.5. With an abuse of notation, we let v denote the trivial extension to a $(\mathsf{d}+1)$-vector field, i.e. the map $(x,z) \mapsto (v(x),0)$. Inserting the identity $F|z|^\gamma - \mathrm{div}(|z|^\gamma\nabla F) = \langle\nabla\rangle^r f\delta_{\mathbb{R}^{\mathsf{d}}\times\{0\}^{\mathsf{k}}}$ and integrating by parts, we see that

$$\begin{aligned}\int_{\mathbb{R}^{\mathsf{d}}} v\cdot\nabla f\langle\nabla\rangle^r f &= \int_{\mathbb{R}^{\mathsf{d}+1}} v\cdot\Big(\frac{1}{2}\nabla(F^2)|z|^\gamma - \nabla F\,\mathrm{div}(|z|^\gamma\nabla F)\Big) \\ &= -\frac{1}{2}\int_{\mathbb{R}^{\mathsf{d}+1}}|z|^\gamma\,\mathrm{div}\,vF^2 + \int_{\mathbb{R}^{\mathsf{d}+1}}|z|^\gamma\nabla v : [\nabla F,\nabla F],\end{aligned} \tag{3.49}$$

where $[\nabla F,\nabla F]$ is the stress-energy tensor associated to ∇F, that is for vector fields $u_1 = (u_1^i), u_2 = (u_2^j)$,

$$[u_1,u_2]_{ij} := u_1^i u_2^j + u_1^j u_2^i - (u_1\cdot u_2)\delta_{ij}, \qquad i,j \in [\mathsf{d}]. \tag{3.50}$$

By Cauchy-Schwarz, the preceding right-hand is $\le$

$$C\|\nabla v\|_{L^\infty}\int_{\mathbb{R}^{\mathsf{d}+1}}|z|^\gamma(F^2 + |\nabla F|^2) = C\|\nabla v\|_{L^\infty}\|f\|^2_{H^{r/2}}. \tag{3.51}$$

where $C = C(\mathsf{d}) > 0$.

As our induction hypothesis, suppose that Proposition 3.1 holds for all $\alpha = 2k + r$, where $k \le m$ and $r \in (0,2]$. We will show that Proposition 3.1 holds for all $\alpha = 2(m+1) + r$, where $r \in (0,2]$.

Writing $\langle\nabla\rangle^{2(m+1)+r} = (I-\Delta)\langle\nabla\rangle^{2m+r}$ and integrating by parts, we find that

$$\begin{aligned}\int_{\mathbb{R}^{\mathsf{d}}} v\cdot\nabla f\langle\nabla\rangle^{2(m+1)+r}f &= \int_{\mathbb{R}^{\mathsf{d}}} v\cdot\nabla f\langle\nabla\rangle^{2m+r}f + \int_{\mathbb{R}^{\mathsf{d}}} v\cdot\nabla\partial_j f\langle\nabla\rangle^{2m+r}\partial_j f\\ &\qquad + \int_{\mathbb{R}^{\mathsf{d}}}\partial_j v\cdot\nabla f\langle\nabla\rangle^{2m+r}\partial_j f,\end{aligned} \tag{3.52}$$

where we follow the convention of Einstein summation. Recall the notation $A_{v,\alpha}$ from (3.2), which we extend in the obvious manner to allow for u to be tensor-valued. By the induction hypothesis, the magnitude of the first term on the right-hand side is $\leq$

$$CA_{v,\alpha-2}\|f\|^2_{H^{m+\frac{r}{2}}} \leq CA_{v,\alpha-2}\|f\|^2_{H^{m+1+\frac{r}{2}}}, \tag{3.53}$$

where the second inequality follows from the trivial embedding $\|\cdot\|_{H^s}\leq\|\cdot\|_{H^{s'}}$ for $s\leq s'$.[7] The induction hypothesis (with f replaced by $\partial_j f$) also implies that the second term on the right-hand side of (3.52) is bounded by

$$CA_{v,\alpha-2}\sum_{j=1}^{\mathsf{d}}\|\partial_j f\|^2_{H^{m+\frac{r}{2}}} \leq CA_{v,\alpha-2}\|f\|^2_{H^{m+1+\frac{r}{2}}}. \tag{3.54}$$

It remains to bound the third term on the right-hand side of (3.52). Expanding $(I-\Delta)^m$ by binomial formula and integrating by parts, we find

$$\begin{aligned}\int_{\mathbb{R}^{\mathsf{d}}}\partial_j v\cdot\nabla f\langle\nabla\rangle^{2m+r}\partial_j f &= \sum_{k=0}^{m}\binom{m}{k}\int_{\mathbb{R}^{\mathsf{d}}}\partial_j v\cdot\nabla f(-\Delta)^k\langle\nabla\rangle^r\partial_j f\\ &= \sum_{k=0}^{m}\binom{m}{k}\int_{\mathbb{R}^{\mathsf{d}}}\langle\nabla\rangle^{r/2}\nabla^{\otimes k}(\partial_j v\cdot\nabla f) : \langle\nabla\rangle^{r/2}\nabla^{\otimes k}\partial_j f.\end{aligned} \tag{3.55}$$

For all $0\leq k\leq m$, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}\left|\int_{\mathbb{R}^{\mathsf{d}}}\langle\nabla\rangle^{r/2}\nabla^{\otimes k}(\partial_j v\cdot\nabla f) : \langle\nabla\rangle^{r/2}\nabla^{\otimes k}\partial_j f\right| &\leq \|\langle\nabla\rangle^{r/2}\nabla^{\otimes k}(\partial_j v\cdot\nabla f)\|_{L^2}\|\langle\nabla\rangle^{r/2}\nabla^{\otimes k}\partial_j f\|_{L^2}\\ &\leq C\tilde{A}_{\nabla v,k,r}\|\nabla f\|_{H^{k+\frac{r}{2}}}\|f\|_{H^{k+1+\frac{r}{2}}}\\ &\leq C\tilde{A}_{\nabla v,k,r}\|f\|^2_{H^{k+1+\frac{r}{2}}},\end{aligned} \tag{3.56}$$

where the second line follows from applying Plancherel's theorem to the second factor and Lemma 3.4 to the first factor on the right-hand side of the first line (recall the notation (3.14)), and the third line follows from another application of Plancherel. Majorizing each factor $\|f\|_{H^{k+1+\frac{r}{2}}}\leq\|f\|_{H^{m+1+\frac{r}{2}}}$, for $0\leq k\leq m$, it follows now that

$$\left|\int_{\mathbb{R}^{\mathsf{d}}}\partial_j v\cdot\nabla f\langle\nabla\rangle^{2m+r}\partial_j f\right| \leq C\|f\|^2_{H^{m+1+\frac{r}{2}}}\sum_{k=0}^{m}\binom{m}{k}\tilde{A}_{\nabla v,k,r}. \tag{3.57}$$

Combining the estimates (3.52), (3.54), (3.57), we obtain that

$$\leq C\|f\|^2_{H^{m+1+\frac{r}{2}}}\left(A_{v,\alpha-2}+\sum_{k=0}^{m}\binom{m}{k}\tilde{A}_{\nabla v,k,r}\right). \tag{3.58}$$

Noting that $\alpha-2 = 2m+r$ and $A_{\cdot,k,r}\leq A_{\cdot,m,r}$ for $k\leq m$ concludes the proof of the induction step, and therefore of the proposition. ☐

[7]Remark that here, the fact that the we are considering the inhomogeneous Sobolev *norm* as opposed to the homogeneous *seminorm* is crucial because otherwise, this embedding is false.

4. Renormalized commutator estimate

In this section, we give the proof of Theorem 1.1. In fact, as advertised in the introduction, we will prove the following more general result that is valid for (s, s)-admissible potentials. The proof proceeds through combining the commutator estimate, valid for regular distributions f, with the potential truncation scheme to handle distributions of the form $f = \frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu$.

Theorem 4.1. *Let g be an (s,a)-admissible potential for $\mathsf{a} \in (\mathsf{d},\mathsf{d}+2)$, and let $\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}} < p \le \infty$. There exists a constant $C>0$ depending only $\mathsf{d},\mathsf{s},\mathsf{a}$, and g through the constants in Proposition 2.9 such that the following holds. Let $\mu \in L^1(\mathbb{R}^\mathsf{d}) \cap L^p(\mathbb{R}^\mathsf{d})$ with $\int_{\mathbb{R}^\mathsf{d}} \mu = 1$. Let $v : \mathbb{R}^\mathsf{d} \to \mathbb{R}^\mathsf{d}$ be a Lipschitz vector field. For any pairwise distinct configuration $X_N \in (\mathbb{R}^\mathsf{d})^N$, it holds that*

$$\left|\int_{(\mathbb{R}^\mathsf{d})^2\setminus\triangle}(v(x)-v(y))\cdot\nabla\mathsf{g}(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}(x,y)\right| \\ \le C\Big(\|\nabla v\|_{L^\infty}+\||\nabla|^{\frac{\mathsf{a}}{2}}v\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}}\Big)\Big(\mathsf{F}_N(X_N,\mu)-\frac{\log\lambda}{2N}\mathbf{1}_{\mathsf{s}=0}+C\|\mu\|_{L^\infty}\lambda^{\mathsf{d}-\mathsf{s}}\Big). \tag{4.1}$$

Proof. Let $\eta>0$, the exact value of which will be specified momentarily. We assume that g is (s,a)-admissible for $\mathsf{d}<\mathsf{a}<\mathsf{d}+2$. Adding and subtracting g_η and applying the triangle inequality, we find

$$\left|\int_{(\mathbb{R}^\mathsf{d})^2\setminus\triangle}(v(x)-v(y))\cdot\nabla\mathsf{g}(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}(x,y)\right| \\ \le \left|\int_{(\mathbb{R}^\mathsf{d})^2}(v(x)-v(y))\cdot\nabla\mathsf{g}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}(x,y)\right| \\ +\left|\int_{(\mathbb{R}^\mathsf{d})^2\setminus\triangle}(v(x)-v(y))\cdot\nabla\mathsf{f}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}\right|. \tag{4.2}$$

Since $|v(x)-v(y)||\nabla\mathsf{g}_\eta(x-y)|$ vanishes along the diagonal by Remark 2.6, we have re-inserted the diagonal in the first integral on the right-hand side above.

As $\int_{(\mathbb{R}^\mathsf{d})^2}\mathsf{g}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}<\infty$, Remark 3.3 and Corollary 3.2 imply that

$$\left|\int_{(\mathbb{R}^\mathsf{d})^2}(v(x)-v(y))\cdot\nabla\mathsf{g}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}(x,y)\right| \\ \le C\Big(\|\nabla v\|_{L^\infty}+\||\nabla|^{\frac{\mathsf{a}}{2}}v\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}}\Big)\int_{(\mathbb{R}^\mathsf{d})^2}\mathsf{g}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}. \tag{4.3}$$

On the other hand, applying the mean-value theorem to $(v(x)-v(y))\cdot\nabla\mathsf{f}_\eta(x-y)$,

$$\left|\int_{(\mathbb{R}^\mathsf{d})^2\setminus\triangle}(v(x)-v(y))\cdot\nabla\mathsf{f}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}\right| \\ \le \|\nabla v\|_{L^\infty}\int_{(\mathbb{R}^\mathsf{d})^2\setminus\triangle}|x-y||\nabla\mathsf{f}_\eta(x-y)|d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}+\mu\Big)^{\otimes 2} \\ \le \|\nabla v\|_{L^\infty}\int_{(\mathbb{R}^\mathsf{d})^2\setminus\triangle}\mathsf{f}_{2\eta}(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}+\mu\Big)^{\otimes 2}. \tag{4.4}$$

where the third line follows from assertion

Applying the estimates (4.3), (4.4) to the right-hand side of (4.2), we arrive at

$$\begin{aligned}(4.5)\quad &\left|\int_{(\mathbb{R}^{\mathsf{d}})^2}(v(x)-v(y))\cdot\nabla\mathsf{g}_\eta(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}(x,y)\right|\\
&\le C\Big(\|\nabla v\|_{L^\infty}+\||\nabla|^{\frac{s}{2}}v\|_{L^{\frac{2\mathsf{d}}{s-2}}}\Big)\Big(\int_{(\mathbb{R}^{\mathsf{d}})^2\setminus\triangle}\mathsf{f}_{\eta/2}(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}+\mu\Big)^{\otimes 2}\\
&\qquad+\int_{(\mathbb{R}^{\mathsf{d}})^2}\mathsf{g}_{\eta/2}(x-y)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu\Big)^{\otimes 2}\Big),\end{aligned}$$

where we have implicitly used that $\mathsf{g}_{\eta/2}-\mathsf{g}_\eta$ has a positive Fourier transform to replace g_η by $\mathsf{g}_{\eta/2}$ in (4.3). Applying Proposition 2.11 to the preceding expression and choosing $\eta=\lambda=(\|\mu\|_{L^\infty}N)^{-1/\mathsf{d}}$, the conclusion of the proof follows after a little bookkeeping. □

5. Application: optimal mean-field convergence rate

This section is devoted to the mean-field limit and the analysis of the mean-field equation.

5.1. Modulated energy bound. Using our main technical result Theorem 1.1, we now show convergence of the empirical measure for the mean-field particle dynamics (1.12) to a (necessarily unique) solution of the limiting PDE (1.15) in the modulated energy distance with the optimal rate $N^{\frac{\mathsf{s}}{\mathsf{d}}-1}$. This proves Theorem 1.3 (cf. [RS23b,]).

Proof of Theorem 1.3. We recall (e.g. see [Ser20, Lemma 2.1] or [RS24b, Lemma 3.6]) that $\mathsf{F}_N(X_N^t,\mu^t)$ satisfies the differential inequality

$$(5.1)\qquad \frac{d}{dt}\mathsf{F}_N(X_N^t,\mu^t)\le\int_{(\mathbb{R}^{\mathsf{d}})^2\setminus\triangle}\nabla\mathsf{g}(x-y)\cdot\big(u^t(x)-u^t(y)\big)d\Big(\frac{1}{N}\sum_{i=1}^N\delta_{x_i^t}-\mu^t\Big)^{\otimes 2}(x,y).$$

where $u^t:=-\mathbb{M}\nabla\mathsf{g}*\mu^t+\mathsf{V}$. Applying the first-order estimate (1.6) of Theorem 1.1 pointwise in t to the preceding right-hand side, then integrating with respect to time both sides of the resulting inequality and applying the fundamental theorem of calculus, it follows that

$$\begin{aligned}(5.2)\quad &\mathsf{F}_N(X_N^t,\mu^t)+\frac{\log\big(N\|\mu^t\|_{L^\infty}\big)}{2N\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}+C\|\mu^t\|_{L^\infty}^{\frac{\mathsf{s}}{\mathsf{d}}}N^{\frac{\mathsf{s}}{\mathsf{d}}-1}\le\mathsf{F}_N(X_N^0,\mu^0)+\frac{\log\big(N\|\mu^t\|_{L^\infty}\big)}{2N\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}\\
&+C\|\mu^t\|_{L^\infty}^{\frac{\mathsf{s}}{\mathsf{d}}}N^{\frac{\mathsf{s}}{\mathsf{d}}-1}+C\int_0^t\|\nabla u^\tau\|_{L^\infty}\Big(\mathsf{F}_N(X_N^\tau,\mu^\tau)+\frac{\log(N\|\mu^\tau\|_{L^\infty})}{2N\mathsf{d}}\mathbf{1}_{\mathsf{s}=0}+\frac{C(N\|\mu^\tau\|_{L^\infty})^{\frac{\mathsf{s}}{\mathsf{d}}}}{N}\Big)d\tau\end{aligned}$$

for some constant $C>0$ depending only on $|\mathbb{M}|_\infty,\mathsf{s},\mathsf{d}$. Assuming that the constant $C>0$ above is sufficiently large depending on d,s, the left-hand side defines a nonnegative quantity in view of Remark 2.12. An application of the Grönwall-Bellman lemma then completes the proof. □

5.2. Mean-field regularity. Suppose that $\mu^t\in\mathcal{P}(\mathbb{R}^{\mathsf{d}})$ is a solution We conclude the paper by verifying that for exact sub-Coulomb Riesz potentials, if $v^t=\mathbb{M}\nabla\mathsf{g}*\mu^t$, then the condition

$$(5.3)\qquad \sup_{t\in[0,T]}\Big(\|\nabla v^t\|_{L^\infty}+\||\nabla|^{\frac{\mathsf{a}}{2}}v^t\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}}\Big)<\infty,$$

for $\mathsf{a}\in(\mathsf{d},\mathsf{d}+2)$, is satisfied provided that the initial density μ° is sufficiently regular. We only consider the exact Riesz case—again for reasons of simplicity—and leave it to the reader to generalize the results of this subsection to the Riesz-type case. Moreover, we only consider the sub-Coulomb case because the Coulomb/super-Coulomb case only features $\|\nabla v^t\|_{L^\infty}$, the control of which has already been established, e.g. see [?]. The addition of an external field V poses no additional complications, only adds extra terms to the computation (cf. Remark 1.5).

We focus here only on the *a priori* estimates: existence of such solutions follows by standard vanishing viscosity arguments (e.g. see [BIK15, Sections 5,6]), while uniqueness follows from Cauchy-Lipschitz theory. Throughout this section, we assume that μ^t is a smooth, probability density-valued solution of the mean-field equation.

The satisfaction of the condition (5.3) is a consequence of the following two lemmas and the nonincrease of $\|\mu^t\|_{L^r}$ for any $1 \le r \le \infty$. For the latter fact, see the proof of [RS23b, Remark 3.4].

Lemma 5.1. *Let $v := \mathbb{M}\nabla \mathsf{g} * f$ for any Schwartz function f. Then it holds that for any $p > \frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}$*

$$\|\nabla v\|_{L^\infty} \le C|\mathbb{M}|_\infty \|f\|_{L^1}^{\frac{p(\mathsf{d}-\mathsf{s}-2)-\mathsf{d}}{\mathsf{d}(p-1)}} \|f\|_{L^p}^{\frac{p(\mathsf{s}+2)}{\mathsf{d}(p-1)}} \tag{5.4}$$

and further supposing $\mathsf{d} \ge 3$,

$$\||\nabla|^{\frac{\mathsf{a}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}} \le C \begin{cases} \|f\|_{L^{\frac{\mathsf{d}}{\mathsf{d}-2-\mathsf{s}}}}, & \mathsf{s} \le \mathsf{d}-1-\frac{\mathsf{a}}{2} \\ \||\nabla|^\alpha f\|_{L^{\frac{\mathsf{d}}{\alpha+\mathsf{d}-2-\mathsf{s}}}}, & \mathsf{s} > \mathsf{d}-1-\frac{\mathsf{a}}{2} \end{cases}, \tag{5.5}$$

where $|\cdot|_\infty$ denotes the operator norm and $C>0$ depends only on $\mathsf{d}, \mathsf{s}, \alpha$.

Lemma 5.2. *Let $\alpha \ge 1$ and $1 < p < \infty$. Then there exists a nondecreasing function $\mathbf{W} : [0,\infty)^3 \to [0,\infty)$, depending only on $\mathsf{d}, \mathsf{s}, \alpha, p$, such that*[8]

$$\forall T \ge 0, \qquad \sup_{t\in[0,T]} \||\nabla|^\alpha \mu^t\|_{L^p} \le \mathbf{W}(T, \|\mu^0\|_{L^\infty}, \||\nabla|^\alpha \mu^0\|_{L^p}). \tag{5.6}$$

Remark that the assumption $\alpha \ge 1$ in the statement is necessary for the argument to close. We begin with the proof of Lemma 5.1, which is a consequence of some elementary harmonic analysis.

Proof of Lemma 5.1. Observe that for any $R > 0$,

$$\begin{aligned} \|\nabla v\|_{L^\infty} &\le C|\mathbb{M}|_\infty \int_{\mathbb{R}^\mathsf{d}} |x-y|^{2+\mathsf{s}-\mathsf{d}} f(y) \\ &\le C|\mathbb{M}|_\infty \int_{\mathbb{R}^\mathsf{d}} |x-y|^{2+\mathsf{s}-\mathsf{d}} f(y) \\ &\le C|\mathbb{M}|_\infty \left(\int_{|x-y|\le R} |x-y|^{2+\mathsf{s}-\mathsf{d}} f(y) + \int_{|x-y|>R} |x-y|^{2+\mathsf{s}-\mathsf{d}} f(y) \right) \\ &\le C|\mathbb{M}|_\infty \Big(\|f\|_{L^p} R^{\frac{\mathsf{d}(p-1)}{p}+2+\mathsf{s}-\mathsf{d}} + R^{2+\mathsf{s}-\mathsf{d}} \|f\|_{L^1} \Big), \end{aligned} \tag{5.7}$$

where the ultimate line is by Hölder's inequality. Implicitly, we are using that $\mathsf{s} < \mathsf{d}-2$. Optimizing the choice of R then yields (5.4).

Recalling that $\frac{1}{\mathsf{c}_{\mathsf{d},\mathsf{s}}}\mathsf{g}$ is the convolution kernel of the Fourier multiplier $|\nabla|^{\mathsf{s}-\mathsf{d}}$, we see that

$$\||\nabla|^{\frac{\mathsf{a}}{2}} v\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}} = \mathsf{c}_{\mathsf{d},\mathsf{s}} \||\nabla|^{\frac{\mathsf{a}}{2}+\mathsf{s}-\mathsf{d}} \mathbb{M}\nabla\mu\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}} \le C\mathsf{c}_{\mathsf{d},\mathsf{s}} |\mathbb{M}|_\infty \||\nabla|^{\frac{\mathsf{a}}{2}+1+\mathsf{s}-\mathsf{d}} \mu\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}}, \tag{5.8}$$

where we have also used the boundedness of the Riesz transform on $L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}$ to replace the operator $\mathbb{M}\nabla$ by $|\mathbb{M}|_\infty |\nabla|$. Note that since we restrict to $\mathsf{d} \ge 3$ and $\mathsf{a} < \mathsf{d}+2$, we always have $2 < \frac{2\mathsf{d}}{\mathsf{a}-2} < \infty$. The operator $|\nabla|^{\frac{\mathsf{a}}{2}+1+\mathsf{s}-\mathsf{d}}$ is smoothing if $\mathsf{s} < \mathsf{d}-1-\frac{\mathsf{a}}{2}$, the identity if $\mathsf{s} = \mathsf{d}-1-\frac{\mathsf{a}}{2}$, and differentiating if $\mathsf{s} > \mathsf{d}-1-\frac{\mathsf{a}}{2}$. We dispense with the first two cases, which are easy.

If $\mathsf{s} \le \mathsf{d}-1-\frac{\mathsf{a}}{2}$ (the equality case is trivial), then the Hardy-Littlewood-Sobolev lemma implies that

$$\||\nabla|^{\frac{\mathsf{a}}{2}+1+\mathsf{s}-\mathsf{d}} f\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}} \le C\|f\|_{L^{\frac{\mathsf{d}}{\mathsf{d}-2-\mathsf{s}}}}. \tag{5.9}$$

[8] In the course of proving the lemma, we show a more precise estimate. We choose to omit it here to simplify the exposition. Also, one could consider $p \in \{1,\infty\}$ for integer α, replacing $|\nabla|^\alpha$ by $\nabla^{\otimes\alpha}$.

If $\mathsf{s} > \mathsf{d} - 1 - \frac{\mathsf{a}}{2}$, then Sobolev embedding implies that for any $\mathsf{s} + 2 > \alpha \geq \frac{\mathsf{a}}{2} + 1 + \mathsf{s} - \mathsf{d}$,

$$\||\nabla|^{\frac{\mathsf{a}}{2}+1+\mathsf{s}-\mathsf{d}} f\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}} \leq C\||\nabla|^{\alpha} f\|_{L^p}, \qquad p = \frac{\mathsf{d}}{\alpha + \mathsf{d} - 2 - \mathsf{s}}. \tag{5.10}$$

□

Remark 5.3. Evidently, we may allow for $\alpha > \mathsf{s} + 2$ (the case $\alpha = \mathsf{s} + 2$ is excluded due to the failure of the endpoint Sobolev embedding), provided that we replace the homogeneous seminorm on the right-hand side with the inhomogeneous norm $\|f\|_{W^{\alpha,1}}$.

We now prove Lemma 5.2.

Proof of Lemma 5.2. Observe from the chain rule and mean-field equation (1.15) that

$$\begin{aligned}
\frac{d}{dt}\||\nabla|^{\alpha}\mu^t\|_{L^p}^p &= p\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t|\nabla|^{\alpha}\operatorname{div}\Big(\mu^t\mathbb{M}\nabla\mathsf{g}*\mu^t\Big) \\
&= p\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t\nabla|\nabla|^{\alpha}\mu^t\cdot\mathbb{M}\nabla\mathsf{g}*\mu^t \\
&\quad + p\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t\nabla\mu^t\cdot|\nabla|^{\alpha}\mathbb{M}\nabla\mathsf{g}*\mu^t \\
&\quad - p\alpha\mathbf{1}_{\alpha\geq1}\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t\nabla^{\otimes2}|\nabla|^{\alpha-2}\mu^t : \nabla\mathbb{M}\nabla\mathsf{g}*\mu^t \\
&\quad + p\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t|\nabla|^{\alpha}\mu^t\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t \\
&\quad + p\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t\mu^t\operatorname{div}|\nabla|^{\alpha}\mathbb{M}\nabla\mathsf{g}*\mu^t \\
&\quad - p\alpha\mathbf{1}_{\alpha\geq1}\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t\nabla|\nabla|^{\alpha-2}\mu^t\cdot\nabla\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t \\
&\quad + \int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t\Big(\mathcal{C}_1^t + \mathcal{C}_2^t\Big),
\end{aligned} \tag{5.11}$$

where we have abbreviated the (higher-order) commutators

$$\begin{aligned}
\mathcal{C}_1^t &\coloneqq |\nabla|^{\alpha}\Big(\nabla\mu^t\cdot\mathbb{M}\nabla\mathsf{g}*\mu^t\Big) - \nabla|\nabla|^{\alpha}\mu^t\cdot\mathbb{M}\nabla\mathsf{g}*\mu^t - \nabla\mu^t\cdot|\nabla|^{\alpha}\mathbb{M}\nabla\mathsf{g}*\mu^t \\
&\quad + \alpha\mathbf{1}_{\alpha\geq1}\nabla^{\otimes2}|\nabla|^{\alpha-2}\mu^t : \nabla\mathbb{M}\nabla\mathsf{g}*\mu^t,
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\mathcal{C}_2^t &\coloneqq |\nabla|^{\alpha}\Big(\mu^t\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t\Big) - |\nabla|^{\alpha}\mu^t\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t - \mu^t|\nabla|^{\alpha}\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t \\
&\quad + \alpha\mathbf{1}_{\alpha\geq1}\nabla|\nabla|^{\alpha-2}\mu^t\cdot\nabla\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t.
\end{aligned} \tag{5.13}$$

Remark that if $\mathbb{M}$ is antisymmetric, then all terms involving $\operatorname{div}\mathbb{M}\nabla$ vanish. Integrating by parts, we see that

$$\begin{aligned}
p\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t\nabla|\nabla|^{\alpha}\mu^t\cdot\mathbb{M}\nabla\mathsf{g}*\mu^t + p\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t|\nabla|^{\alpha}\mu^t\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t \\
= (p-1)\int_{\mathbb{R}^\mathsf{d}} ||\nabla|^{\alpha}\mu^t|^p\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t.
\end{aligned} \tag{5.14}$$

By our assumption (1.13) on $\mathbb{M}$ and the explicit form of g, we have that $\operatorname{div}\mathbb{M}\nabla\mathsf{g} \leq 0$. As $\mu^t \geq 0$, it follows that the preceding right-hand side is nonpositive and may be discarded. We estimate directly the remaining terms.

Assuming that $\alpha \geq 1$ (otherwise, the estimate will not close due to the $\nabla\mu^t$ factor below), Hölder's inequality yields for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$,

$$\begin{aligned}\int_{\mathbb{R}^{\mathsf{d}}} ||\nabla|^\alpha\mu^t|^{p-2}|\nabla|^\alpha\mu^t\nabla\mu^t\cdot|\nabla|^\alpha\mathbb{M}\nabla\mathsf{g}*\mu^t &\leq \||\nabla|^\alpha\mu^t\|_{L^p}^{p-1}\|\nabla\mu^t\|_{L^{p_1}}\||\nabla|^\alpha\mathbb{M}\nabla\mathsf{g}*\mu^t\|_{L^{p_2}}\\ &\leq C|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^{p-1}\|\nabla\mu^t\|_{L^{p_1}}\||\nabla|^{\alpha+1+\mathsf{s}-\mathsf{d}}\mu^t\|_{L^{p_2}}.\end{aligned} \tag{5.15}$$

If $\alpha+1+\mathsf{s}-\mathsf{d}\leq 0$, then we choose $(p_1,p_2)=(p,\infty)$ and Gagliardo-Nirenberg interpolation [**?**, Theorem 2.44] plus Hölder's inequality imply

$$\|\nabla\mu^t\|_{L^{p_1}}\||\nabla|^{\alpha+1+\mathsf{s}-\mathsf{d}}\mu^t\|_{L^{p_2}} \leq C\|\mu^t\|_{L^p}^{1-\frac{1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}. \tag{5.16}$$

If $\alpha+1+\mathsf{s}-\mathsf{d}>0$, then we choose $(p_1,p_2)=(\alpha p,\frac{\alpha p}{\alpha-1})$ so that by Gagliardo-Nirenberg interpolation,

$$\begin{aligned}\|\nabla\mu^t\|_{L^{p_1}}\||\nabla|^{\alpha+1+\mathsf{s}-\mathsf{d}}\mu^t\|_{L^{p_2}} &\leq C\|\mu^t\|_{L^\infty}^{\frac{\alpha-1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\mathsf{d}-1-\mathsf{s})}{\mathsf{d}-2-\mathsf{s}}}}^{\frac{\mathsf{d}-1-\mathsf{s}}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha+1+\mathsf{s}-\mathsf{d}}{\alpha}}\\ &= C\|\mu^t\|_{L^\infty}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\mathsf{d}-1-\mathsf{s})}{\mathsf{d}-2-\mathsf{s}}}}^{\frac{\mathsf{d}-1-\mathsf{s}}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha+2+\mathsf{s}-\mathsf{d}}{\alpha}}.\end{aligned} \tag{5.17}$$

Thus, in all cases,

$$\begin{aligned}\int_{\mathbb{R}^{\mathsf{d}}} ||\nabla|^\alpha\mu^t|^{p-2}|\nabla|^\alpha\mu^t\nabla\mu^t\cdot|\nabla|^\alpha\mathbb{M}\nabla\mathsf{g}*\mu^t &\leq C|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^{p-1}\times\\ &\Big(\|\mu^t\|_{L^p}^{1-\frac{1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\mathbf{1}_{\alpha+1+\mathsf{s}-\mathsf{d}\leq 0}\\ &\quad+\|\mu^t\|_{L^\infty}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\mathsf{d}-1-\mathsf{s})}{\mathsf{d}-2-\mathsf{s}}}}^{\frac{\mathsf{d}-1-\mathsf{s}}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha+2+\mathsf{s}-\mathsf{d}}{\alpha}}\Big).\end{aligned} \tag{5.18}$$

Arguing similarly, using also the L^p boundedness of the Riesz transform,

$$\begin{aligned}&\int_{\mathbb{R}^{\mathsf{d}}} ||\nabla|^\alpha\mu^t|^{p-2}|\nabla|^\alpha\mu^t\nabla^{\otimes 2}|\nabla|^{\alpha-2}\mu^t:\nabla\mathbb{M}\nabla\mathsf{g}*\mu^t\\ &\leq \||\nabla|^\alpha\mu^t\|_{L^p}^{p-1}\|\nabla^{\otimes 2}|\nabla|^{\alpha-2}\mu^t\|_{L^p}\|\nabla\mathbb{M}\nabla\mathsf{g}*\mu^t\|_{L^\infty}\\ &\leq C|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^p\||\nabla|^{2+\mathsf{s}-\mathsf{d}}\mu^t\|_{L^\infty}\\ &\leq C'|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^p\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}},\end{aligned} \tag{5.19}$$

where we have also used Hölder's inequality on the L^∞ factor in obtaining the final line. Implicitly, we are using the assumption $\mathsf{s}<\mathsf{d}-2$.

Again using Hölder's inequality, we find that

$$\int_{\mathbb{R}^{\mathsf{d}}} ||\nabla|^\alpha\mu^t|^{p-2}|\nabla|^\alpha\mu^t\mu^t\operatorname{div}|\nabla|^\alpha\mathbb{M}\nabla\mathsf{g}*\mu^t \leq \||\nabla|^\alpha\mu^t\|_{L^p}^{p-1}\|\mu^t\|_{L^{p_1}}\|\operatorname{div}|\nabla|^\alpha\mathbb{M}\nabla\mathsf{g}*\mu^t\|_{L^{p_2}}, \tag{5.20}$$

where $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$. We choose p_2 so that

$$\mathsf{d}-\mathsf{s}-2=\mathsf{d}\Big(\frac{1}{p}-\frac{1}{p_2}\Big)\Rightarrow p_2=\frac{\mathsf{d}p}{\mathsf{d}-p(\mathsf{d}-\mathsf{s}-2)}, \tag{5.21}$$

which, by the Hardy-Littlewood-Sobolev lemma, implies that

$$\|\operatorname{div}|\nabla|^\alpha\mathbb{M}\nabla\mathsf{g}*\mu^t\|_{L^{p_2}} \leq C|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}. \tag{5.22}$$

Therefore,

$$\int_{\mathbb{R}^{\mathsf{d}}} ||\nabla|^\alpha\mu^t|^{p-2}|\nabla|^\alpha\mu^t\mu^t\operatorname{div}|\nabla|^\alpha\mathbb{M}\nabla\mathsf{g}*\mu^t \leq C|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^p\|\mu^t\|_{L^{\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}}}. \tag{5.23}$$

Next, we use Hölder's inequality to bound

$$\begin{aligned}&\int_{\mathbb{R}^{\mathsf{d}}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t\nabla|\nabla|^{\alpha-2}\mu^t\cdot\nabla\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t\\ &\le \||\nabla|^{\alpha}\mu^t\|_{L^p}^{p-1}\||\nabla|^{\alpha-1}\mu^t\|_{L^{p_1}}\|\nabla\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t\|_{L^{p_2}}\\ &\le C|\mathbb{M}|_{\infty}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{p-1}\||\nabla|^{\alpha-1}\mu^t\|_{L^{p_1}}\||\nabla|^{3+\mathsf{s}-\mathsf{d}}\mu^t\|_{L^{p_2}},\end{aligned} \tag{5.24}$$

where $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$. If $3+\mathsf{s}-\mathsf{d}\le 0$, then we may choose $(p_1,p_2)=(p,\infty)$ and use Gagliardo-Nirenberg interpolation plus Hölder's inequality (similar to (5.16)) to estimate

$$\||\nabla|^{\alpha-1}\mu^t\|_{L^{p_1}}\||\nabla|^{3+\mathsf{s}-\mathsf{d}}\mu^t\|_{L^{p_2}}\le C\|\mu^t\|_{L^p}^{\frac{1}{\alpha}}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}. \tag{5.25}$$

If $3+\mathsf{s}-\mathsf{d}>0$, then (similar to (5.17)) we choose $(p_1,p_2)=(\frac{\alpha p}{p-1},\alpha p)$ and use Gagliardo-Nirenberg interpolation to obtain

$$\||\nabla|^{\alpha-1}\mu^t\|_{L^{p_1}}\||\nabla|^{3+\mathsf{s}-\mathsf{d}}\mu^t\|_{L^{p_2}}\le \|\mu^t\|_{L^\infty}^{\frac{1}{\alpha}}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\alpha+\mathsf{d}-3-\mathsf{s})}{\mathsf{d}-\mathsf{s}-2}}}^{\frac{\alpha+\mathsf{d}-3-\mathsf{s}}{\alpha}}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{\frac{3+\mathsf{s}-\mathsf{d}}{\alpha}}. \tag{5.26}$$

Thus, in all cases, we have

$$\begin{aligned}&\int_{\mathbb{R}^{\mathsf{d}}} ||\nabla|^{\alpha}\mu^t|^{p-2}|\nabla|^{\alpha}\mu^t\nabla|\nabla|^{\alpha-2}\mu^t\cdot\nabla\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t\le C|\mathbb{M}|_{\infty}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{p-\frac{1}{\alpha}}\Big(\\ &\|\mu^t\|_{L^p}^{\frac{1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\mathbf{1}_{3+\mathsf{s}-\mathsf{d}\le 0}+\|\mu^t\|_{L^\infty}^{\frac{1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\alpha+\mathsf{d}-3-\mathsf{s})}{\mathsf{d}-\mathsf{s}-2}}}^{\frac{\alpha+\mathsf{d}-3-\mathsf{s}}{\alpha}}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{\frac{3+\mathsf{s}-\mathsf{d}}{\alpha}}\mathbf{1}_{3+\mathsf{s}-\mathsf{d}>0}\Big).\end{aligned} \tag{5.27}$$

Finally, we handle the commutator terms. Using the Kato-Ponce commutator inequality [Li19, Theorem 5.1], we find that

$$\|\mathcal{C}_1^t\|_{L^p}\le C\Big(\|\nabla\mu^t\|_{L^{p_1}}\||\nabla|^{\alpha}\mathbb{M}\nabla\mathsf{g}*\mu^t\|_{L^{p_2}}+\mathbf{1}_{\alpha\ge 2}\|\nabla|\nabla|^{\alpha-1}\mu^t\|_{L^{p_3}}\|\nabla\mathbb{M}\nabla\mathsf{g}*\mu^t\|_{L^{p_4}}\Big), \tag{5.28}$$

where $C>0$ depends only on d,s,p. Here, $1<p_1,p_2,p_3,p_4\le\infty$ are such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}=\frac{1}{p}$. Each of the terms on the right-hand side have been estimated above, and we find that

$$\begin{aligned}\|\mathcal{C}_1^t\|_{L^p}&\le C|\mathbb{M}|_{\infty}\Big(\|\mu^t\|_{L^p}^{1-\frac{1}{\alpha}}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{\frac{1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\mathbf{1}_{\alpha+1+\mathsf{s}-\mathsf{d}\le 0}\\ &+\|\mu^t\|_{L^\infty}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\mathsf{d}-1-\mathsf{s})}{\mathsf{d}-2-\mathsf{s}}}}^{\frac{\mathsf{d}-1-\mathsf{s}}{\alpha}}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{\frac{\alpha+2+\mathsf{s}-\mathsf{d}}{\alpha}}\mathbf{1}_{\alpha+1+\mathsf{s}-\mathsf{d}>0}+\||\nabla|^{\alpha}\mu^t\|_{L^p}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}}\mathbf{1}_{\alpha\ge 2}\Big).\end{aligned} \tag{5.29}$$

Similarly,

$$\|\mathcal{C}_2^t\|_{L^p}\le C\Big(\|\mu^t\|_{L^{p_1}}\|\operatorname{div}|\nabla|^{\alpha}\mathbb{M}\nabla\mathsf{g}*\mu^t\|_{L^{p_2}}+\mathbf{1}_{\alpha\ge 2}\||\nabla|^{\alpha-1}\mu^t\|_{L^{p_3}}\|\nabla\operatorname{div}\mathbb{M}\nabla\mathsf{g}*\mu^t\|_{L^{p_4}}\Big) \tag{5.30}$$

with the p_i as above. Again, we note that each of the terms on the right-hand side has been previously estimated, and we find that

$$\begin{aligned}\|\mathcal{C}_2^t\|_{L^p}&\le C|\mathbb{M}|_{\infty}\Big(\|\mu^t\|_{L^{\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}}}\||\nabla|^{\alpha}\mu^t\|_{L^p}+\|\mu^t\|_{L^p}^{\frac{1}{\alpha}}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\mathbf{1}_{\substack{3+\mathsf{s}-\mathsf{d}\le 0\\ \alpha\ge 2}}\\ &\qquad+\|\mu^t\|_{L^\infty}^{\frac{1}{\alpha}}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\alpha+\mathsf{d}-3-\mathsf{s})}{\mathsf{d}-\mathsf{s}-2}}}^{\frac{\alpha+\mathsf{d}-3-\mathsf{s}}{\alpha}}\||\nabla|^{\alpha}\mu^t\|_{L^p}^{\frac{3+\mathsf{s}-\mathsf{d}}{\alpha}}\mathbf{1}_{\substack{3+\mathsf{s}-\mathsf{d}>0\\ \alpha\ge 2}}\Big).\end{aligned} \tag{5.31}$$

Combining the preceding estimates (5.29), (5.31) with Hölder's and triangle inequalities, we conclude that

$$
\begin{aligned}
(5.32)\quad &\int_{\mathbb{R}^{\mathsf{d}}} ||\nabla|^\alpha \mu^t|^{p-2}|\nabla|^\alpha \mu^t\Big(\mathcal{C}_1^t+\mathcal{C}_2^t\Big)\\
&\le C|\mathbb{M}|_\infty \||\nabla|^\alpha\mu^t\|_{L^p}^{p-1}\Big(\|\mu^t\|_{L^p}^{1-\frac{1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\mathbf{1}_{\alpha+1+\mathsf{s}-\mathsf{d}\le 0}\\
&+\|\mu^t\|_{L^\infty}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\mathsf{d}-1-\mathsf{s})}{\mathsf{d}-2-\mathsf{s}}}}^{\frac{\mathsf{d}-1-\mathsf{s}}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha+2+\mathsf{s}-\mathsf{d}}{\alpha}}\mathbf{1}_{\alpha+1+\mathsf{s}-\mathsf{d}>0}+\||\nabla|^\alpha\mu^t\|_{L^p}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}}\mathbf{1}_{\alpha\ge 2}\\
&+\|\mu^t\|_{L^{\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}}}\||\nabla|^\alpha\mu^t\|_{L^p}+\|\mu^t\|_{L^p}^{\frac{1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\mathbf{1}_{\substack{3+\mathsf{s}-\mathsf{d}\le 0\\ \alpha\ge 2}}\\
&+\|\mu^t\|_{L^\infty}^{\frac{1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\alpha+\mathsf{d}-3-\mathsf{s})}{\mathsf{d}-\mathsf{s}-2}}}^{\frac{\alpha+\mathsf{d}-3-\mathsf{s}}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{3+\mathsf{s}-\mathsf{d}}{\alpha}}\mathbf{1}_{\substack{3+\mathsf{s}-\mathsf{d}>0\\ \alpha\ge 2}}\Big).
\end{aligned}
$$

Collecting the estimates (5.14), (5.18), (5.19), (5.23), (5.27), (5.32), we conclude that

$$
\begin{aligned}
(5.33)\quad &\||\nabla|^\alpha\mu^t\|_{L^{\frac{2\mathsf{d}}{\mathsf{a}-2}}}\le C|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^{p-1}\times\\
&\Big(\|\mu^t\|_{L^p}^{1-\frac{1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\mathbf{1}_{\alpha+1+\mathsf{s}-\mathsf{d}\le 0}\\
&+\|\mu^t\|_{L^\infty}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\mathsf{d}-1-\mathsf{s})}{\mathsf{d}-2-\mathsf{s}}}}^{\frac{\mathsf{d}-1-\mathsf{s}}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha+2+\mathsf{s}-\mathsf{d}}{\alpha}}\Big)\\
&+C'|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^{p}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}}\\
&+C|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^{p}\|\mu^t\|_{L^{\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}}}\\
&+C|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^{p-\frac{1}{\alpha}}\Big(\\
&\|\mu^t\|_{L^p}^{\frac{1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\mathbf{1}_{3+\mathsf{s}-\mathsf{d}\le 0}+\|\mu^t\|_{L^\infty}^{\frac{1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\alpha+\mathsf{d}-3-\mathsf{s})}{\mathsf{d}-\mathsf{s}-2}}}^{\frac{\alpha+\mathsf{d}-3-\mathsf{s}}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{3+\mathsf{s}-\mathsf{d}}{\alpha}}\mathbf{1}_{3+\mathsf{s}-\mathsf{d}>0}\Big)\\
&+C|\mathbb{M}|_\infty\||\nabla|^\alpha\mu^t\|_{L^p}^{p-1}\Big(\|\mu^t\|_{L^p}^{1-\frac{1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{(\alpha+1+\mathsf{s})}{\mathsf{d}}}\mathbf{1}_{\alpha+1+\mathsf{s}-\mathsf{d}\le 0}\\
&+\|\mu^t\|_{L^\infty}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\mathsf{d}-1-\mathsf{s})}{\mathsf{d}-2-\mathsf{s}}}}^{\frac{\mathsf{d}-1-\mathsf{s}}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha+2+\mathsf{s}-\mathsf{d}}{\alpha}}\mathbf{1}_{\alpha+1+\mathsf{s}-\mathsf{d}>0}+\||\nabla|^\alpha\mu^t\|_{L^p}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-2}{\mathsf{d}}}\mathbf{1}_{\alpha\ge 2}\\
&+\|\mu^t\|_{L^{\frac{\mathsf{d}}{\mathsf{d}-\mathsf{s}-2}}}\||\nabla|^\alpha\mu^t\|_{L^p}+\|\mu^t\|_{L^p}^{\frac{1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^1}^{1-\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\|\mu^t\|_{L^\infty}^{\frac{\mathsf{d}-\mathsf{s}-3}{\mathsf{d}}}\mathbf{1}_{\substack{3+\mathsf{s}-\mathsf{d}\le 0\\ \alpha\ge 2}}\\
&+\|\mu^t\|_{L^\infty}^{\frac{1}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{\alpha-1}{\alpha}}\|\mu^t\|_{L^{\frac{p(\alpha+\mathsf{d}-3-\mathsf{s})}{\mathsf{d}-\mathsf{s}-2}}}^{\frac{\alpha+\mathsf{d}-3-\mathsf{s}}{\alpha}}\||\nabla|^\alpha\mu^t\|_{L^p}^{\frac{3+\mathsf{s}-\mathsf{d}}{\alpha}}\mathbf{1}_{\substack{3+\mathsf{s}-\mathsf{d}>0\\ \alpha\ge 2}}\Big),
\end{aligned}
$$

where $C>0$ depends only on $\mathsf{d},\mathsf{s},p,\alpha$. Noting that the exponent of the $\||\nabla|^\alpha\mu^t\|_{L^p}$ factor is $\le p$, the desired conclusion now follows from Grönwall's inequality and the nonincrease of $\|\mu^t\|_{L^r}$ for any $1\le r\le\infty$, and using interpolation to control all the factors $\|\mu^t\|_{L^r}$ in terms of $\|\mu^t\|_{L^1}$ and $\|\mu^t\|_{L^\infty}$. $\square$

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