

21-106 Co-Calculus Homework 2 Solutions

Find all real zeros of the following polynomials. Write each polynomial in factored form.

6. $f(x) = x^3 - 6x^2 + 11x - 6$

Solution:

First of all, since f is a polynomial of degree 3, we know that f has at most 3 real zeros. Second, the coefficients in front of the powers of x in $f(x)$ change sign 3 times ($1 \rightarrow -6 \rightarrow 11 \rightarrow -6$), so $f(x)$ has either 3 positive real zeros or $3-2=1$ positive real zero. Note that

$$\begin{aligned} f(-x) &= (-x)^3 - 6(-x)^2 + 11(-x) - 6 \\ &= -x^3 - 6x^2 - 11x - 6, \end{aligned}$$

so we see that the coefficients of $f(-x)$ have NO sign changes, so $f(x)$ has no negative real zeros.

To find the potential rational zeros of f , we use the Rational zeros theorem, which says that, if x is a zero of f , then it must be of the form $\frac{p}{q}$, where p is a factor of the coefficient in front of $x^0 = 1$, and q is a factor of the coefficient in front of x^n , where n is the highest power of x in the polynomial. In this case, possibilities for p include 1, -1, 2, -2, 3, -3, 6, or -6, and possibilities for q include 1 and -1. So possible RATIONAL zeros of f include 1, -1, 2, -2, 3, -3, 6, or -6. Let's test $x = 1$. Plugging $x = 1$ into $f(x)$ gives $f(1) = 1 - 6 + 11 - 6 = 0$. Great news! This means 1 is a zero of f , so $x - 1$ is a factor of f . So $f(x) = (x - 1)q(x)$, where $q(x)$ is a polynomial. To solve for $q(x)$, we use long division:

$$\begin{array}{r} x^2 - 5x + 6 \\ x - 1 \overline{) x^3 - 6x^2 + 11x - 6} \\ \underline{-(x^3 - x^2)} \\ 0 - 5x^2 + 11x \\ \underline{-(-5x^2 + 5x)} \\ 0 + 6x - 6 \\ \underline{-(6x - 6)} \\ 0. \end{array}$$

This means $f(x) = (x - 1)(x^2 - 5x + 6)$. To find other zeros of f , we can just find the other factors of f , and to do this, we simply factor $(x^2 - 5x + 6)$. We get $(x^2 - 5x + 6) = (x - 2)(x - 3)$ (you can use the quadratic formula to do this). So, finally,

$$f(x) = (x - 1)(x - 2)(x - 3),$$

and the zeros of f are given by 1, 2, and 3. Again this follows because $(x - 1)$, $(x - 2)$, and $(x - 3)$ are factors of f !

9. $f(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$

Solution:

First of all, since f is a polynomial of degree 4, it has at most 4 real zeros. Since the coefficients of $f(x)$ change sign 4 times, f has either 4, $4-2=2$, or $4-4=0$ positive real zeros; that is, f has either 4, 2, or 0 positive real zeros (recall it is the number of sign changes or the number of sign changes minus an even integer). To determine the possible number of negative real zeros, we write

$$\begin{aligned} f(-x) &= (-x)^4 - 10(-x)^3 + 35(-x)^2 - 50(-x) + 24 \\ &= x^4 + 10x^3 + 35x^2 + 50x + 24. \end{aligned}$$

Since coefficients of $f(-x)$ never change sign, we conclude that f has NO negative real zeros.

Using the Rational zeros theorem, we list the potential rational zeros of f . Using the same strategy as in problem 5 above, we get that potential rational zeros are: 1, -1, 2, -1, 3, -3, 4, -4, 6, -6, 8, -8, 12, -12, 24, -24. We can eliminate all negative numbers from the list since f has no negative real zeros, so possible rational zeros of f include 1, 2, 3, 4, 6, 8, 12, and 24. We test $x = 1$, and we find that $f(1) = 0$. So $(x - 1)$ is a factor of f , which means $f(x) = (x - 1)q(x)$, where $q(x)$ is a polynomial. To determine $q(x)$, we use long division (which I leave to you), and we get $f(x) = (x - 1)(x^3 - 9x^2 + 26x - 24)$. Unfortunately, we cannot factor $q(x) = x^3 - 9x^2 + 26x - 24$ in our heads (at least I can't), so we need to factor $q(x)$ by determining its zeros. Note that the rational zeros for q must come from the list of potential rational zeros for our original polynomial f , because if $x - c$ is a factor of $q(x)$, then it is also a factor of $f(x)$ (meaning if c is a zero of q , then it is also a zero of f). So we go back to our list and test $x = 1$ AGAIN to see if $x = 1$ is a zero of $q(x)$ (it's possible that $x - 1$ is a factor that shows up twice in $f(x)$). In this case $q(1) \neq 0$. Luckily, $q(2) = 0$, so $x - 2$ is a factor of $q(x)$. Now, $q(x) = (x - 2)r(x)$, where $r(x)$ is a polynomial. To determine $r(x)$, we use long division, and we get $r(x) = x^2 - 7x + 12$, and $q(x) = (x - 2)(x^2 - 7x + 12)$, so $f(x) = (x - 1)q(x) = (x - 1)(x - 2)(x^2 - 7x + 12)$.

To finish finding zeros of f and factoring f , we just need to factor $x^2 - 7x + 12 = (x - 4)(x - 3)$. So

$$(1) \quad f(x) = (x - 1)(x - 2)(x - 4)(x - 3)$$

and the zeros of f are 1, 2, 3, and 4.

11. $f(x) = 9x^6 + 44x^4 + 31x^2 - 4$

Solution:

First of all, f is a polynomial of degree 6, so it has at most 6 real zeros. Using Descartes rule of signs, we see only one sign change in coefficients of $f(x)$, so f has

exactly one positive real zero. Note that

$$\begin{aligned} f(-x) &= 9(-x)^6 + 44(-x)^4 + 31(-x)^2 - 4 \\ &= 9x^6 + 44x^4 + 31x^2 - 4 \end{aligned}$$

which, again, has only one sign change, so f has exactly one NEGATIVE real zero. This is very useful information, because now we know that f has exactly two real zeros: one positive and one negative. Once we find them, we will know that we can stop working. Also, since $f(x) = f(-x)$, we can conclude that if $x = c$ is a zero of $f(x)$, then $x = -c$ is a zero of $f(x)$, so we really only need to find one zero of f . Using the rational zeros theorem, we see that potential rational zeros include 1, -1, 2, -2, 4, -4, $\frac{1}{9}$, $-\frac{1}{9}$, $\frac{1}{3}$, $-\frac{1}{3}$, $\frac{2}{3}$, $-\frac{2}{3}$, $\frac{4}{3}$, $-\frac{4}{3}$, $\frac{4}{9}$, $-\frac{4}{9}$, $\frac{2}{9}$, $-\frac{2}{9}$. Certainly, 1, -1, 2, -2, 4, and -4 will not work, because when you plug them in, you will get a large number minus 4. After some testing, we find that $\frac{1}{3}$ works, and by the logic above, $-\frac{1}{3}$ works as well. We have found all of the real zeros. We know by the factor theorem that $x - \frac{1}{3}$ and $x + \frac{1}{3}$ are factors of f , that is,

$$f(x) = (x - \frac{1}{3})(x + \frac{1}{3})q(x)$$

where $q(x)$ cannot be factored (since f has no other real zeros). To find $q(x)$, we use long division to divide $(x - \frac{1}{3})(x + \frac{1}{3}) = x^2 - \frac{1}{9}$ into $f(x)$. We get $q(x) = 9x^4 + 45x^2 + 36$. So, finally,

$$f(x) = (x - \frac{1}{3})(x + \frac{1}{3})(9x^4 + 45x^2 + 36).$$

We cannot factor f any further. The real zeros of f are $x = \frac{1}{3}$ and $-\frac{1}{3}$.