Calculus I, 21-111
Test 2
March 30

Name: Solutions

Recitation (circle yours): Klobusicky (TR 2:30) Mogin (TR 3:30)

Note: No calculators or electronic devices of any sort are allowed. Show all work necessary to obtain your answers. No credit will be given for answers without justification.

1. (20 points) Let $f(x) = 3x\sqrt{x^2+9}, g(x) = \frac{3x-1}{2x+1}, h(x) = x(x^2+1)^4$. Give a simplified expression for each of the following:

   (a) $\frac{d}{dx}(h(x)) = \frac{d}{dx}\left(x(x^2+1)^4\right)$

   $$= x \cdot 4(x^2+1)^3 \cdot 2x + (x^2+1)^4 \cdot 1$$

   $$= 8x^2(x^2+1)^3 + (x^2+1)^4$$

   (b) $f'(0)$ $\quad f'(x) = 3x\frac{1}{2}(x^2+9)^{-1/2} \cdot 2x + 3\sqrt{x^2+9}$

   $$= \frac{3x^2}{\sqrt{x^2+9}} + 3\sqrt{x^2+9}$$

   $$f'(0) = \frac{3 \cdot 0^2}{\sqrt{0^2+9}} + 3\sqrt{0^2+9} = 0 \cdot 3 \cdot 3 = \boxed{9}$$

   (c) $g'(2)$

   $$g'(x) = \frac{(2x+1) \cdot 3 - (3x-1) \cdot 2}{(2x+1)^2}$$

   $$= \frac{6x+3-6x+2}{(2x+1)^2} = \frac{5}{(2x+1)^2}$$

   $$g'(2) = \frac{5}{(2 \cdot 2+1)^2} = \frac{5}{5^2} = \boxed{\frac{1}{5}}$$

2. (20 points) Sketch the curve $y = x^4 - \frac{4}{3}x^3$, labeling clearly all maximums, minimums, and points of inflection.

$$y' = 4x^3 - 4x^2 = 4x^2(x-1) = 0 \text{ if } x = 0, 1$$

		0		1	
x^2	+		+		+
$x-1$	−		−		+
y'	−		−	min	+

$$y'' = 12x^2 - 8x = 4x(3x-2) = 0 \text{ if } x = 0, \ x = \frac{2}{3}$$

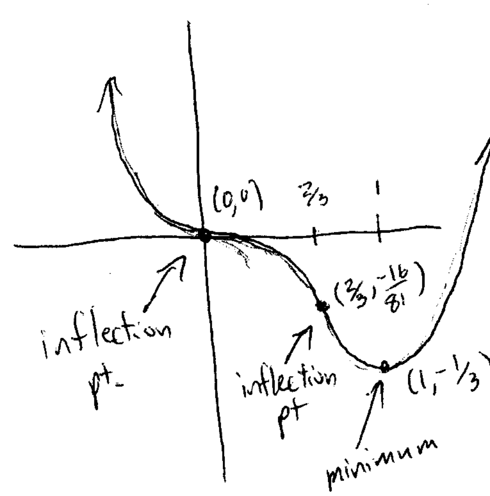
		0		2/3	
x	−		+		+
$3x-2$	−		−		+
y''	+	inf. pt	−	inf. pt	+

$y(0) = 0$

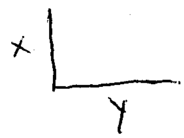
$y(1) = 1 - \frac{4}{3} = -\frac{1}{3}$

$y\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^4 - \frac{4}{3}\left(\frac{2}{3}\right)^3 = \frac{2^4 - 4 \cdot 2^3}{3^4} = \frac{-16}{81}$

		0		2/3		1	
y'	−		−		−		+
y''	+		−		+		+
y							



3. (20 points) A long rectangular sheet of metal 40 inches wide is to be made into a gutter by turning up strips vertically along the two sides. How many inches should be turned up on each side in order to maximize the amount of water the gutter can carry?



Maximize $f = xy$

subject to $2x + y = 40$

$y = 40 - 2x$

$$f = x(40 - 2x)$$
$$= 40x - 2x^2$$

$f' = 40 - 4x = 0$ if $x = 10$

$f'' = -4 < 0$ for all x, so f concave down everywhere

$\therefore$ $x = 10$ is an abs. max.

$\boxed{\text{10 inches should be turned up on each side}}$

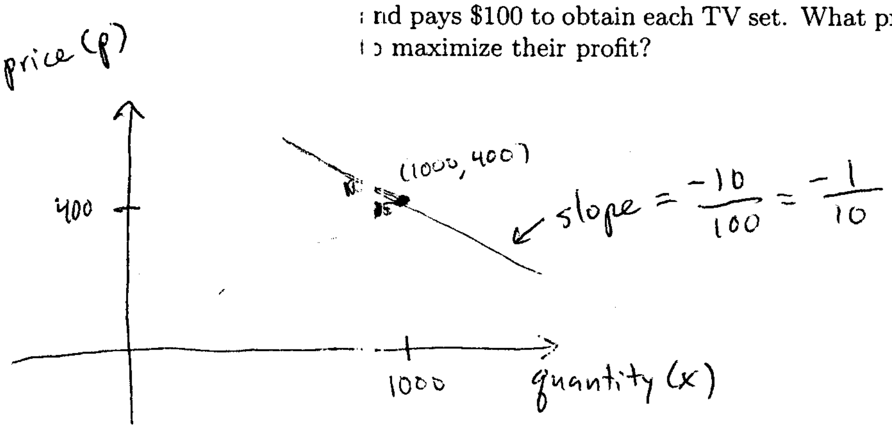
$(x = 10,\ y = 20)$

4. (20 points) An electronics retailer sells 1000 television sets a week while charging \$400 per set. They estimate that for every \$10 they take off the price, they will sell 100 more sets per week.

   (a) Find the demand function, assuming it is linear.

   (b) What price should they charge to maximize ~~profit~~ revenue?

   (c) Assume the retailer has a fixed cost of \$10,000 per week in overhead, and pays \$100 to obtain each TV set. What price should they charge to maximize their profit?



a) demand equation: $p - 400 = \frac{-1}{10}(x - 1000)$

$p - 400 = -\frac{x}{10} + 100$

$p = -\frac{x}{10} + 500$

b) Revenue $= x \cdot \text{price} = x\left(-\frac{x}{10} + 500\right) = \frac{-x^2}{10} + 500x$

$R' = -\frac{2x}{10} + 500 = -\frac{x}{5} + 500 = 0$ if $x = 2500$

so concave down: abs max $\rightarrow R'' = -\frac{1}{5} < 0$

So they should charge $p = -\frac{2500}{10} + 500 = -250 + 500 = \boxed{\$250}$

c) Cost $= 10\,000 + 100x$

Profit $= R(x) - C(x) = \frac{-x^2}{10} + 500x - (10{,}000 + 100x)$

$= -\frac{x^2}{10} + 400x - 10{,}000$

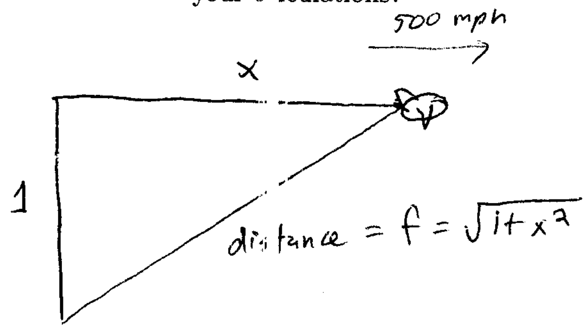
$P' = \frac{-x}{5} + 400 = 0$ if $x = 2000$

$P'' = -\frac{1}{5} < 0$ so $x = 2000$ is abs. max.

They should charge $p = \frac{-2000}{10} + 500 = -200 + 500 = \boxed{\$300}$

5. (20 pc nts) A plane flying horizontally at an altitude of one mile and a speed of 500 mph passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when the plane is 2 miles away from the station.

(optional: 1 point bonus) How would you expect this rate to change as the plane gets further away from the station? Can you justify this using your calculations?



given $\frac{dx}{dt} = 500$

find $\frac{df}{dt}$ when $f = 2$

$f = \sqrt{1+x^2}$

$\frac{df}{dt} = \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x \frac{dx}{dt} = \frac{x}{\sqrt{1+x^2}} \frac{dx}{dt}$

When $f = 2$, find x: $\sqrt{1+x^2} = 2$

$1+x^2 = 4$

$x^2 = 3$

$x = \sqrt{3}$

$\left.\frac{df}{dt}\right|_{f=2} = \frac{\sqrt{3}}{\sqrt{1+\sqrt{3}^2}} \cdot 500$

$= \frac{\sqrt{3}}{\sqrt{4}} \cdot 500 = \frac{\sqrt{3} \cdot 500}{2} = \sqrt{3} \cdot 250$ mph

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Optional: $\frac{df}{dt} = \frac{x}{\sqrt{1+x^2}} \frac{dx}{dt}$. As x gets big, $\frac{x}{\sqrt{x^2+1}} \to 1$,

so $\frac{df}{dt} \to 500$ (when the plane is far away it's basically flying directly away from the radar station at a speed of 500 mph)