HW 7 solutions

Section 2.5

#10) max. $Q = xyz$

subject to $x+y=1$

$y+z=2$

$x,y,z \geq 0$

$x = 1-y$, $z = 2-y$ $\longrightarrow$ $Q = (1-y)y(2-y)$

$= (y-y^2)(2-y)$

$= y^3 - 3y^2 + 2y$

$Q'(y) = 3y^2 - 6y + 2 = 0$

$y = \dfrac{6 \pm \sqrt{36-4\cdot 3\cdot 2}}{2\cdot 3} = 1 \pm \dfrac{\sqrt{12}}{6} = 1 \pm \dfrac{2\sqrt{3}}{6} = 1 \pm \dfrac{\sqrt{3}}{3}$

$Q''(y) = 6y - 6$

so $Q''(1+\frac{\sqrt{3}}{3}) > 0$ (min)

$Q''(1-\frac{\sqrt{3}}{3}) < 0$ (max)

check endpoint: $y=0 \Rightarrow Q=0$

$y=1 \Rightarrow x=0 \Rightarrow Q=0$

$y = 1-\frac{\sqrt{3}}{3} \Rightarrow x = \frac{\sqrt{3}}{3}$, $z = 1+\frac{\sqrt{3}}{3}$

$Q = (1-\frac{\sqrt{3}}{3})(\frac{\sqrt{3}}{3})(1+\frac{\sqrt{3}}{3}) = \frac{2\sqrt{3}}{9}$ maximum.

#22) min $x+y$

s.t. $xy = 100 \rightarrow y = \frac{100}{x}$

$x,y \geq 0$

minimize $f(x) = x + \frac{100}{x}$

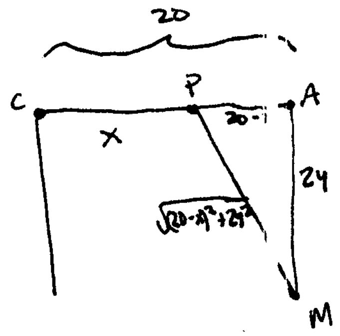
$f'(x) = 1 - \frac{100}{x^2} = 0$ if $x = \pm 10$ (ignore -10)

$f''(x) = 1 + \frac{200}{x^3} > 0$ at $x=10$, so it's a max.

$f'' > 0$ for all $x > 0$, so $x=10$ is an absolute max.

So if $x=10$, then $y=10$, and the minimum sum is $10+10=20$.

#27)



$$\text{Cost} = 6(\text{distance } CP) + 10(\text{distance } PM)$$
$$= 6x + 10\sqrt{(20-x)^2+24^2}$$
$$= 6x + 10\sqrt{x^2-40x+976}$$

$$\text{Cost}' = 6 + 10\tfrac{1}{2}(x^2-40x+976)^{-1/2}(2x-40)$$
$$= 6 + \frac{5(2x-40)}{\sqrt{x^2-40x+976}}$$

$$\cancel{2x-40=0}$$
$$\cancel{2x=40}$$
$$\cancel{x=20}$$

$$=0 \text{ if } \frac{5(2x-40)}{\sqrt{x^2-40x+976}} = -6$$

$$5(2x-40) = 6\sqrt{x^2-40x+976}$$
$$25(2x-40)^2 = 36(x^2-40x+976)$$
$$25(4x^2-160x+1600) = 36(x^2-40x+976)$$
$$100x^2-4000x+40{,}000 = 36x^2-1440x+35{,}136$$
$$64x^2-2560x+4864=0$$
$$64(x^2-40x+76)=0$$
$$64(x-38)(x-2)=0$$

if $x=38$ or $x=2$

ignore $x=38$ (not feasible)

Verify $x=2$ is a min:

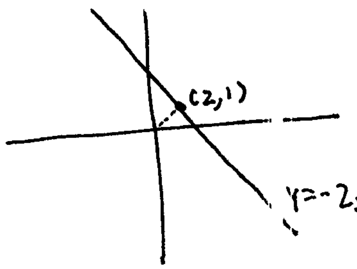
$\text{Cost}'(3) > 0$

$\text{Cost}'(1) < 0$

so c': $-$ | 2 | $+$ $\Rightarrow x=2$ is a max.

Minimum cost $= 6\cdot 2 + 10\sqrt{2^2-40\cdot 2+976} = 312$

#31)



$\min \sqrt{x^2+y^2}$

s.t. $y=-2x+5$

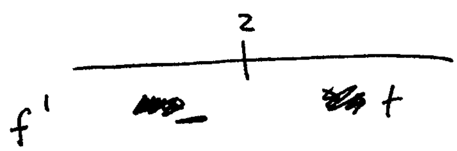
$\min \sqrt{x^2+(-2x+5)^2} = f(x)$

$f'(x) = \frac{1}{2}\left(x^2+(-2x+5)^2\right)^{-1/2}\left(2x+2(-2x+5)(-2)\right)$

$= \dfrac{2x+8x-20}{2\sqrt{x^2+(-2x+5)^2}}$

$= \dfrac{5x-10}{\sqrt{x^2+(-2x+5)^2}}$

$=0$ if $5x-10=0$ $\quad x=2$



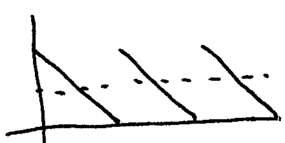
so $x=2$ is a min

$y=-2\cdot 2+5=1$

$\Rightarrow$ $(2,1)$ is the closest point.

Section 2.6

#8)



r orders of x books each

cost = ordering cost + carrying cost

$= 40r + 2\left(\frac{x}{2}\right) = 40r + x$

subject to $rx = 8000$

Minimize $40r + \dfrac{8000}{r} = f(r)$

$f'(r) = 40 - \dfrac{8000}{r^2} = 0 \qquad \dfrac{8000}{r^2} = 40$

$8000 = 40r^2$

$200 = r^2$

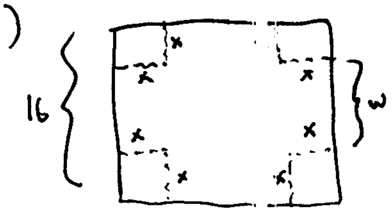
$10\sqrt{2} = r$

Verify $10\sqrt{2}$ is a min:

$f''(r) = \dfrac{16{,}000}{r^3} > 0$ if $r>0$ so $10\sqrt{2}$ is an absolute min.

They should place $10\sqrt{2} \approx 14$ orders.

#21)



maximize $w^2 x$ (volume)

subject to $w + 2x = 16$

$w = 16 - 2x$

maximize $f(x) = (16-2x)^2 x$

$= (256 - 64x + 4x^2)\, x$

$= 4(x^3 - 16x^2 + 64x)$

$f'(x) = 4(3x^2 - 32x + 64)$

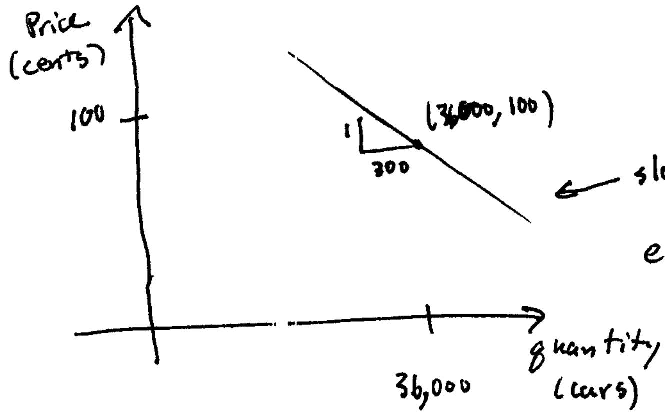
$= 4(3x-8)(x-8) = 0$ if $x = \frac{8}{3}$ or $x = 8$

		$8/3$		8	
$3x-8$	$-$		$+$		$+$
$x-8$	$-$		$-$		$+$
f'	$+$	max	$-$	min	$+$

So $\boxed{x = \frac{8}{3}}$ is a max in the interval $0 \le x \le 8$

Section 2.7

#16) Demand function:



slope $= -\dfrac{1}{300}$

equation of line: $p - 100 = \dfrac{-1}{300}(x - 36{,}000)$

$$p = \frac{-x}{300} + 220$$

Revenue $= px$

$$p = \frac{-x}{300} + 220$$

So, maximize $R(x) = \left(-\dfrac{x}{300} + 220\right)x = \dfrac{-x^2}{300} + 220x$

$$R' = \frac{-2x}{300} + 220 = 0$$

if $220 = \dfrac{2x}{300}$

$$33{,}000 = x$$

$R'' = -\dfrac{2}{300} < 0$ so concave down everywhere

$\Rightarrow x = 33{,}000$ abs. max

They should charge $p(33{,}000) = \dfrac{-33{,}000}{300} + 220$

$= 110$ cents.

#17) $x = 1000$: of kw-hrs

Demand equation: $p(x) = 60 - 10^{-5}x$

Cost $C(x) = 7\cdot 10^6 + 30x$

Revenue $R(x) = xp(x) = x(60 - 10^{-5}x) = 60x - 10^{-5}x^2$

Profit $P(x) = R(x) - C(x)$

$$= 60x - 10^{-5}x^2 - (7\cdot 10^6 + 30x)$$

$$= -10^{-5}x^2 + 30x - 7\cdot 10^6$$

a) Maximize profit:

$$P'(x) = -2\cdot 10^{-5}x + 30 \quad = 0$$

if $\quad 2\cdot 10^{-5}x = 30$

$$x = 1.5\cdot 10^6$$

$P'' = -2\cdot 10^{-5} < 0$ so P concave down everywhere

$\Rightarrow x = 1.5\cdot 10^6$ abs. max.

The should charge $p(6\cdot 10^6) = 60 - 10^{-5}(1.5\cdot 10^{-6})$

$$= 60 - 15 = 45 \text{ dollars.}$$

b) $C_1(x) = 7\cdot 10^6 + 40x$

$P_1(x) = -10^{-5}x^2 + 20x - 7\cdot 16^6$

$P_1'(x) = -2\cdot 10^{-5}x + 20 \quad = 0$ if $\quad 2\cdot 10^{-5}x = 20$

$x = 10^6$ (max same as part a)

price $= p(10^6) = 60 - 10^{-5}(10^6) = 50$ dollars

No. To maximize profit, the price should only be raised by \$5.

Section 3.1

#1) $y' = (x+1)(3x^2+5) + (x^3+5x+2)(1)$

$= 3x^3 + 3x^2 + 5x + 5 + x^3 + 5x + 2$

$= 4x^3 + 3x^2 + 10x + 7$

#4) $y' = (x^2+3)(2) + (2x-1)(2x)$

$= 2x^2 + 6 + 4x^2 - 2x$

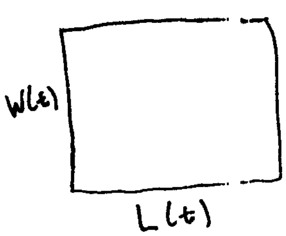
$= 6x^2 - 2x + 6$

#14) $y' = \dfrac{(x+1)\cdot 0 - 1(1)}{(x+1)^2} + \dfrac{(x-1)(0) - 1(1)}{(x-1)^2}$

$= \dfrac{1}{(x+1)^2} + \dfrac{1}{(x-1)^2}$

#42) $h'(x) = (x^2+2x-1)f'(x) + f(x)(2x+2)$

#51)



$A(t) = W(t)\,L(t)$

$A' = W(t)L'(t) + L(t)W'(t)$

$= 5\cdot 4 + 6\cdot 3 = 38$

when $W=5$, $L=6$.

The area is increasing at a rate of 38 in^2/sec.

Section 3.2

#2) $f(g(x)) = \left(\frac{1}{x+1}\right) - 1 = \frac{1}{x+1} - \frac{x+1}{x+1} = \frac{-x}{x+1}$

#4) $f(g(x)) = \frac{(x+3)+1}{(x+3)-3} = \frac{x+4}{x}$

#6) $h(x) = (9x^2+2x-5)^7 = f(g(x))$

where $f(x) = x^7$

$g(x) = 9x^2+2x-5$

#8) $h(x) = (5x^2+1)^{-1/2} = f(g(x))$

where $f(x) = x^{-1/2}$

$g(x) = 5x^2-1$

#12) $y' = 10(x^4+x^2)(4x^3+2x)$

$= 10(4x^7+2x^5+4x^5+2x^3) = 40x^7+60x^5+20x^3$

#14) $y' = 5x^3 \cdot 4(2-x)^3(-1) + (2-x)^4 \cdot 15x^2$

$= -20x^3(2-x)^3 + 15(2-x)^4x^2$

#22) $h'(x) = 2f'(2x+1)\cdot 2$

$= 4f'(2x+1)$

#30) $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

$= \frac{1}{2}g(x)^{-1/2}\cdot 2x$

$= \frac{1}{\cancel{2}}(x^2+1)^{-1/2}\cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}$

#38) $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

$= \frac{1}{2}(u+1)^{-1/2} \cdot 4x$

$= \frac{2}{\sqrt{2x^2+1}}$

#42) $\left.\frac{dy}{dt}\right|_{t=1} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$= 2(x^2-2) \cdot 2x \cdot \left(-1(t+1)^{-2}\right) \Big|_{t=1}$

$= \left.\frac{-4(x^2-2)x}{(t+1)^2}\right|_{t=1}$ $\left(t=1 \Rightarrow x = \frac{1}{t+1} = \frac{1}{1+1} = \frac{1}{2}\right)$

$= \frac{-4\left(\frac{1}{2}^2 - 2\right)\frac{1}{2}}{(1+1)^2}$

$= \frac{-\cancel{4}\left(-\frac{7}{\cancel{4}}\right)\frac{1}{2}}{4}$

$= \frac{7}{8}$