Our goal

Let $\Delta_n$ denote the regular $n$-simplex.
Our goal

Let $\Delta_n$ denote the regular $n$-simplex.

Main question

How may we choose a 1-codimensional hyperplane $H$ passing through the center of $\Delta_n$, so that the volume of the intersection $\text{vol}_{n-1}(\Delta_n \cap H)$ is minimized?
Motivation

If $K$ is a convex body, we call a set of the form $K \cap H$ (where $H$ is a 1-codimensional hyperplane) a section of $K$. 

Bourgain's slicing problem

Does every convex body $K$ of volume 1 admit a section whose volume is at least some universal constant, independent of the dimension $n$?
Motivation

If $K$ is a convex body, we call a set of the form $K \cap H$ (where $H$ is a 1-codimensional hyperplane) a section of $K$. If $H$ passes through the barycenter of $K$, we call it a central section.
Motivation

If $K$ is a convex body, we call a set of the form $K \cap H$ (where $H$ is a 1-codimensional hyperplane) a section of $K$. If $H$ passes through the barycenter of $K$, we call it a central section.

Bourgain’s slicing problem
Does every convex body $K$ of volume 1 admit a section whose volume is at least some universal constant, independent of the dimension $n$?
Motivation

If $K$ is a convex body, we call a set of the form $K \cap H$ (where $H$ is a 1-codimensional hyperplane) a \textit{section} of $K$. If $H$ passes through the barycenter of $K$, we call it a \textit{central section}.

Bourgain's slicing problem

Does every convex body $K$ of volume 1 admit a section whose volume is at least some universal constant, independent of the dimension $n$?

- Open problem
Motivation

If $K$ is a convex body, we call a set of the form $K \cap H$ (where $H$ is a 1-codimensional hyperplane) a *section* of $K$. If $H$ passes through the barycenter of $K$, we call it a *central section*.

**Bourgain’s slicing problem**

Does every convex body $K$ of volume 1 admit a section whose volume is at least some universal constant, independent of the dimension $n$?

- Open problem
- Key to understanding the uniform distribution on a high-dimensional convex body
Motivation

If $K$ is a convex body, we call a set of the form $K \cap H$ (where $H$ is a 1-codimensional hyperplane) a section of $K$. If $H$ passes through the barycenter of $K$, we call it a central section.

Bourgain’s slicing problem

Does every convex body $K$ of volume 1 admit a section whose volume is at least some universal constant, independent of the dimension $n$?

- Open problem
- Key to understanding the uniform distribution on a high-dimensional convex body
- Connections to isoperimetry in high dimensions (cf. KLS conjecture)
Previous work

A general type of question
Given a specific convex body $K$, can we identify its minimum central section?

---


2 Douglas Hensley. “Slicing the Cube in $\mathbb{R}^n$ and Probability (Bounds for the Measure of a Central Cube Slice in $\mathbb{R}^n$ by Probability Methods)”. In: *Proceedings of the American Mathematical Society* 73.1 (1979), pp. 95–100.


A general type of question

Given a specific convex body $K$, can we identify its minimum central section? Maximum central section?

---

1. Hadwiger, “Gitterperiodische Punktmengen und Isoperimetrie”.
2. Hensley, “Slicing the Cube in $\mathbb{R}^n$ and Probability (Bounds for the Measure of a Central Cube Slice in $\mathbb{R}^n$ by Probability Methods)”.  
3. Ball, “Cube slicing in $\mathbb{R}^n$”.
4. Webb, “Central slices of the regular simplex”. 
Previous work

A general type of question

Given a specific convex body $K$, can we identify its minimum central section? Maximum central section?

- $K = Q_n$ ($n$-dimensional hypercube), minimal central section identified in [Hadwiger 1972$^1$, Hensley 1979$^2$]

---

$^1$Hadwiger, “Gitterperiodische Punktmengen und Isoperimetrie”.

$^2$Hensley, “Slicing the Cube in $\mathbb{R}^n$ and Probability (Bounds for the Measure of a Central Cube Slice in $\mathbb{R}^n$ by Probability Methods)”.

$^3$Ball, “Cube slicing in $\mathbb{R}^n$”.

$^4$Webb, “Central slices of the regular simplex”.

A general type of question

Given a specific convex body $K$, can we identify its minimum central section? Maximum central section?

- $K = Q_n$ ($n$-dimensional hypercube), minimal central section identified in [Hadwiger 1972\textsuperscript{1}, Hensley 1979\textsuperscript{2}]
- $K = Q_n$, maximal central section identified in [Ball 1986\textsuperscript{3}]

---

\textsuperscript{1} Hadwiger, “Gitterperiodische Punktmengen und Isoperimetrie”.
\textsuperscript{2} Hensley, “Slicing the Cube in $\mathbb{R}^n$ and Probability (Bounds for the Measure of a Central Cube Slice in $\mathbb{R}^n$ by Probability Methods)”.
\textsuperscript{3} Ball, “Cube slicing in $\mathbb{R}^n$”.
\textsuperscript{4} Webb, “Central slices of the regular simplex”.
Previous work

A general type of question

Given a specific convex body $K$, can we identify its minimum central section? Maximum central section?

- $K = Q_n$ ($n$-dimensional hypercube), minimal central section identified in [Hadwiger 1972$^1$, Hensley 1979$^2$]
- $K = Q_n$, maximal central section identified in [Ball 1986$^3$]
- $K = \Delta_n$ ($n$-dimensional regular simplex), maximal central section identified in [Webb 1996$^4$]

---

$^1$ Hadwiger, “Gitterperiodische Punktmengen und Isoperimetrie”.

$^2$ Hensley, “Slicing the Cube in $\mathbb{R}^n$ and Probability (Bounds for the Measure of a Central Cube Slice in $\mathbb{R}^n$ by Probability Methods)”.

$^3$ Ball, “Cube slicing in $\mathbb{R}^n$”.

$^4$ Webb, “Central slices of the regular simplex”.
Simplex minimum

This leaves open the question from the beginning:

**Simplex minimum**

What is the minimum central section of the regular simplex?

---

Simplex minimum

This leaves open the question from the beginning:

**Simplex minimum**

What is the minimum central section of the regular simplex?

**Conjecture**

The minimum central section is the central section $\Delta_n \cap H$ facet that’s parallel to a facet.

---

5 Brzezinski, “Volume estimates for sections of certain convex bodies”.
Simplex minimum

This leaves open the question from the beginning:

**Simplex minimum**

What is the minimum central section of the regular simplex?

**Conjecture**

The minimum central section is the central section $\Delta_n \cap H_{\text{facet}}$ that's parallel to a facet.

**Previous best bound** [Brzezinski 2013$^5$]

The central section $\Delta_n \cap H_{\text{facet}}$ is within a factor of $\frac{2\sqrt{3}}{e} \approx 1.27$ of the minimum.

---

$^5$Brzezinski, “Volume estimates for sections of certain convex bodies”.
Main result

Conjecture is true up to a $1 - o(1)$ factor [T. 2024$^6$]

The central section $\Delta_n \cap H_{\text{facet}}$ is within a factor of $1 - o(1)$ of the minimum. (Little $o$ is with respect to the dimension $n$.)

---

Main result

Conjecture is true up to a $1 - o(1)$ factor [T. 2024$^6$]

The central section $\Delta_n \cap H_{\text{facet}}$ is within a factor of $1 - o(1)$ of the minimum. (Little $o$ is with respect to the dimension $n$.)

Tools used:

---

$^6$Tang, “Simplex slicing: an asymptotically-sharp lower bound”.
Main result

Conjecture is true up to a $1 - o(1)$ factor [T. 2024$^6$]

The central section $\Delta_n \cap H_{\text{facet}}$ is within a factor of $1 - o(1)$ of the minimum. (Little $o$ is with respect to the dimension $n$.)

Tools used:

- probability distributions

---

$^6$Tang, “Simplex slicing: an asymptotically-sharp lower bound”.
Main result

Conjecture is true up to a $1 - o(1)$ factor [T. 2024\textsuperscript{6}]

The central section $\Delta_n \cap H_{\text{facet}}$ is within a factor of $1 - o(1)$ of the minimum. (Little $o$ is with respect to the dimension $n$.)

Tools used:

- probability distributions
- Fourier analysis

\textsuperscript{6}Tang, “Simplex slicing: an asymptotically-sharp lower bound”. 
Main result

Conjecture is true up to a $1 - o(1)$ factor [T. 2024$^6$]

The central section $\Delta_n \cap H_{\text{facet}}$ is within a factor of $1 - o(1)$ of the minimum. (Little $o$ is with respect to the dimension $n$.)

Tools used:

- *probability distributions*
- *Fourier analysis*
- New: *moving the contour of integration of a meromorphic function*

---

$^6$Tang, “Simplex slicing: an asymptotically-sharp lower bound”.

Main result

Conjecture is true up to a $1 - o(1)$ factor [T. 2024$^6$]

The central section $\Delta_n \cap H_{\text{facet}}$ is within a factor of $1 - o(1)$ of the minimum. (Little $o$ is with respect to the dimension $n$.)

Tools used:

- *probability distributions*
- *Fourier analysis*
- New: *moving the contour of integration of a meromorphic function*

We’ll prove this result in the remainder of the presentation.

---

$^6$Tang, “Simplex slicing: an asymptotically-sharp lower bound”. 
Tool: probability distributions

Embed $\Delta_n$ into $\mathbb{R}^{n+1}$ via

$$\Delta_n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \bigg| \begin{array}{c} x_1 + x_2 + \cdots + x_{n+1} = 1 \\ x_i \geq 0 \text{ for each } i \end{array} \right\}.$$
Tool: probability distributions

Embed $\Delta_n$ into $\mathbb{R}^{n+1}$ via

$$\Delta_n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \begin{array}{c}
x_1 + x_2 + \cdots + x_{n+1} = 1 \\
x_i \geq 0 \text{ for each } i
\end{array} \right\}.$$

Central sections $\Delta_n \cap H$ correspond to a choice of vector $a$ with

$$\begin{cases}
a_1 + a_2 + \cdots + a_{n+1} = 0 \\
\sum a_i^2 = 1
\end{cases}$$

where $a$ is the normal vector to $H$. 

Idea: Instead of $\Delta_n$, consider the density $\Phi(x_1, x_2, \ldots, x_{n+1}) = \begin{cases} e^{-x_1-x_2-\cdots-x_{n+1}} & \text{if each } x_i \geq 0 \\
0 & \text{otherwise} \end{cases}$

Then $\int_{a \perp \Phi} dH_n$ is proportional to the volume of the section. Minimum central sections correspond to minimizing $\int_{a \perp \Phi} dH_n$. 

8 / 27
Tool: probability distributions

Embed $\Delta_n$ into $\mathbb{R}^{n+1}$ via

$$\Delta_n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \left| x_1 + x_2 + \cdots + x_{n+1} = 1, x_i \geq 0 \text{ for each } i \right. \right\}.$$

Central sections $\Delta_n \cap H$ correspond to a choice of vector $a$ with

$$\begin{cases} a_1 + a_2 + \cdots + a_{n+1} = 0 \\ a_1^2 + a_2^2 + \cdots + a_{n+1}^2 = 1 \end{cases}$$

where $a$ is the normal vector to $H$.

Idea: Instead of $\Delta_n$, consider the density

$$\Phi(x_1, x_2, \ldots, x_{n+1}) = \begin{cases} e^{-x_1-x_2-\cdots-x_{n+1}} & \text{if each } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $\int_{a \perp} \Phi \, dH_n$ is proportional to the volume of the section.

Minimum central sections correspond to minimizing $\int_{a \perp} \Phi \, dH_n$.  

8 / 27
Tool: probability distributions

Embed $\Delta_n$ into $\mathbb{R}^{n+1}$ via

$$\Delta_n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \left| \begin{array}{c}
x_1 + x_2 + \cdots + x_{n+1} = 1 \\
x_i \geq 0 \text{ for each } i
\end{array} \right. \right\}.$$ 

Central sections $\Delta_n \cap H$ correspond to a choice of vector $a$ with

$$\begin{cases} a_1 + a_2 + \cdots + a_{n+1} = 0 \\ a_1^2 + a_2^2 + \cdots + a_{n+1}^2 = 1 \end{cases}$$

where $a$ is the normal vector to $H$.

Idea: Instead of $\Delta_n$, consider the density

$$\Phi(x_1, x_2, \ldots, x_{n+1}) = \begin{cases} e^{-x_1-x_2-\cdots-x_{n+1}} & \text{if each } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $\int_{a^\perp} \Phi \, d\mathcal{H}^n$ is proportional to the volume of the section.
Embed $\Delta_n$ into $\mathbb{R}^{n+1}$ via

$$\Delta_n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \middle| x_1 + x_2 + \cdots + x_{n+1} = 1, x_i \geq 0 \text{ for each } i \right\}.$$

Central sections $\Delta_n \cap H$ correspond to a choice of vector $a$ with

$$\begin{cases} a_1 + a_2 + \cdots + a_{n+1} = 0 \\ a_1^2 + a_2^2 + \cdots + a_{n+1}^2 = 1 \end{cases}$$

where $a$ is the normal vector to $H$.

Idea: Instead of $\Delta_n$, consider the density

$$\Phi(x_1, x_2, \ldots, x_{n+1}) = \begin{cases} e^{-x_1 - x_2 - \cdots - x_{n+1}} & \text{if each } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $\int_{a_{\perp}} \Phi \, d\mathcal{H}^n$ is proportional to the volume of the section. Minimum central sections correspond to minimizing $\int_{a_{\perp}} \Phi \, d\mathcal{H}^n$. 
But $\Phi$ is a product measure, so $\int_{a^\perp} \Phi \, d\mathcal{H}^n$ is the density at 0 of the random variable

$$Z_a := a_1 Y_1 + a_2 Y_2 + \cdots + a_{n+1} Y_{n+1}$$

(where the $Y_i$ are i.i.d. standard exponentials (mean 1)).
But $\Phi$ is a product measure, so $\int_{a^{-1}} \Phi \, d\mathcal{H}^n$ is the density at 0 of the random variable

$$Z_a := a_1 Y_1 + a_2 Y_2 + \cdots + a_{n+1} Y_{n+1}$$

(where the $Y_i$ are i.i.d. standard exponentials (mean 1)).
But $\Phi$ is a product measure, so $\int_{a \perp} \Phi \, d\mathcal{H}^n$ is the density at 0 of the random variable

$$Z_a := a_1 Y_1 + a_2 Y_2 + \cdots + a_{n+1} Y_{n+1}$$

(where the $Y_i$ are i.i.d. standard exponentials (mean 1)).

Let $G_a(x)$ denote the density of $Z_a$, so what we said above is

$$\int_{a \perp} \Phi \, d\mathcal{H}^n = G_a(0).$$
Tool: probability distributions

Reduction
The minimum central section corresponds to a choice of vector $a$ minimizing $G_a(0)$. 

Conjectured minimizer $a$ facet satisfies $G_a(0) = \sqrt{\frac{n}{n+1}} \approx 1/e$. It's hard to optimize over the set $S_n \cap 1 \perp$ (the feasible region of $a$).

Expand the feasible region:

- Let $u \in S_n$ be arbitrary (the feasible region of $u$ has one fewer constraint than that of $a$).
- Define $Z_u := u^1(Y^1 - 1) + u^2(Y^2 - 1) + \cdots + u^{n+1}(Y^{n+1} - 1)$.

This extends the earlier definition of $Z_a$ since $a^1(Y^1 - 1) + a^2(Y^2 - 1) + \cdots + a^{n+1}(Y^{n+1} - 1) = a^1Y^1 + a^2Y^2 + \cdots + a^{n+1}Y^{n+1}$. 

Reduction

The minimum central section corresponds to a choice of vector $a$ minimizing $G_a(0)$. Conjectured minimizer $a_{\text{facet}}$ satisfies

$$G_{a_{\text{facet}}}(0) = \sqrt{\frac{n}{n+1}} \left( \frac{n}{n+1} \right)^{n-1} \approx \frac{1}{e}.$$
Reduction

The minimum central section corresponds to a choice of vector $a$ minimizing $G_a(0)$. Conjectured minimizer $a_{\text{facet}}$ satisfies

$$G_{a_{\text{facet}}}(0) = \sqrt{\frac{n}{n+1}} \left( \frac{n}{n+1} \right)^{n-1} \approx \frac{1}{e}.$$  

It’s hard to optimize over the set $S^n \cap \mathbf{1}^\perp$ (the feasible region of $a$).
Reduction
The minimum central section corresponds to a choice of vector $a$ minimizing $G_a(0)$. Conjectured minimizer $a_{\text{facet}}$ satisfies

$$G_{a_{\text{facet}}}(0) = \sqrt{\frac{n}{n+1}} \left( \frac{n}{n+1} \right)^{n-1} \approx \frac{1}{e}. $$

It’s hard to optimize over the set $S^n \cap \mathbf{1}^\perp$ (the feasible region of $a$). Expand the feasible region:
Tool: probability distributions

Reduction

The minimum central section corresponds to a choice of vector $a$ minimizing $G_a(0)$. Conjectured minimizer $a_{\text{facet}}$ satisfies

$$G_{a_{\text{facet}}}(0) = \sqrt{\frac{n}{n+1}} \left( \frac{n}{n+1} \right)^{n-1} \approx \frac{1}{e}.\]$$

It’s hard to optimize over the set $S^n \cap 1^\perp$ (the feasible region of $a$). Expand the feasible region:

- Let $u \in S^n$ be arbitrary (the feasible region of $u$ has one fewer constraint than that of $a$).
Tool: probability distributions

Reduction

The minimum central section corresponds to a choice of vector $a$ minimizing $G_a(0)$. Conjectured minimizer $a_{\text{facet}}$ satisfies

$$G_{a_{\text{facet}}}(0) = \sqrt{\frac{n}{n+1}} \left( \frac{n}{n+1} \right)^{n-1} \approx \frac{1}{e}.$$ 

It’s hard to optimize over the set $S^n \cap 1^\perp$ (the feasible region of $a$). Expand the feasible region:

- Let $u \in S^n$ be arbitrary (the feasible region of $u$ has one fewer constraint than that of $a$).
- Define $Z_u := u_1(Y_1 - 1) + u_2(Y_2 - 1) + \cdots + u_{n+1}(Y_{n+1} - 1)$. 

Tool: probability distributions

Reduction

The minimum central section corresponds to a choice of vector \( a \) minimizing \( G_a(0) \). Conjectured minimizer \( a_{\text{facet}} \) satisfies

\[
G_{a_{\text{facet}}}(0) = \sqrt{\frac{n}{n+1}} \left( \frac{n}{n+1} \right)^{n-1} \approx \frac{1}{e}.
\]

It’s hard to optimize over the set \( S^n \cap 1^\perp \) (the feasible region of \( a \)). Expand the feasible region:

- Let \( u \in S^n \) be arbitrary (the feasible region of \( u \) has one fewer constraint than that of \( a \)).
- Define \( Z_u := u_1(Y_1 - 1) + u_2(Y_2 - 1) + \cdots + u_{n+1}(Y_{n+1} - 1) \).
- This extends the earlier definition of \( Z_a \) since

\[
a_1(Y_1 - 1) + a_2(Y_2 - 1) + \cdots + a_{n+1}(Y_{n+1} - 1)
= a_1Y_1 + a_2Y_2 + \cdots + a_{n+1}Y_{n+1} - (a_1 + a_2 + \cdots + a_{n+1})
= a_1Y_1 + a_2Y_2 + \cdots + a_{n+1}Y_{n+1}.
\]
Question
What’s the minimum possible value that $G_u(0)$ can attain, as $u$ varies in $S^n$?
Tool: probability distributions

Question
What’s the minimum possible value that $G_u(0)$ can attain, as $u$ varies in $S^n$?

Our result
$G_u(0) \geq \frac{1}{e}$ for each $u \in S^n$. Equality achieved if $u = (1) \in S^0$. 

We lost a bit by expanding the feasible region from $S^n \cap 1^\perp \ni a$ to $S^n \ni u$. Indeed, the minimum over $u$ of $G_u(0)$ is exactly $\frac{1}{e}$, but we think the minimum over $a$ of $G_a(0)$ is given by $G_a$ facet $(0) = \sqrt{n}/\sqrt{n+1}$.

But certainly $\frac{1}{e} = \min_u G_u(0) \leq \min_a G_a(0) \leq G_a$ facet $(0)$, and since $G_a$ facet $(0) = \frac{1}{e} \left(1 + o(1)\right)$, we lost at most a $1 + o(1)$ factor by expanding the feasible region.
Question
What’s the minimum possible value that $G_u(0)$ can attain, as $u$ varies in $S^n$?

Our result
$G_u(0) \geq \frac{1}{e}$ for each $u \in S^n$. Equality achieved if $u = (1) \in S^0$.

We lost a bit by expanding the feasible region from $S^n \cap \mathbf{1}^\perp \ni a$ to $S^n \ni u$. Indeed, the minimum over $u$ of $G_u(0)$ is exactly $1/e$, but we think the minimum over $a$ of $G_a(0)$ is given by $G_a\text{-facet}(0) = \sqrt{n/(n+1)}$. But certainly $1/e = \min_u G_u(0) \leq \min_a G_a(0) \leq G_a\text{-facet}(0)$, and since $G_a\text{-facet}(0) = 1/e(1 + o(1))$, we lost at most a $1 + o(1)$ factor by expanding the feasible region.
Question
What’s the minimum possible value that $G_u(0)$ can attain, as $u$ varies in $S^n$?

Our result
$G_u(0) \geq \frac{1}{e}$ for each $u \in S^n$. Equality achieved if $u = (1) \in S^0$.

We lost a bit by expanding the feasible region from $S^n \cap \mathbf{1}^\perp \ni a$ to $S^n \ni u$. Indeed, the minimum over $u$ of $G_u(0)$ is exactly $\frac{1}{e}$, but we think the minimum over $a$ of $G_a(0)$ is given by

$$G_{a_{\text{facet}}}(0) = \sqrt{\frac{n}{n+1}} \left( \frac{n}{n+1} \right)^{n-1}.$$
Tool: probability distributions

Question
What’s the minimum possible value that $G_u(0)$ can attain, as $u$ varies in $S^n$?

Our result
$G_u(0) \geq \frac{1}{e}$ for each $u \in S^n$. Equality achieved if $u = (1) \in S^0$.
We lost a bit by expanding the feasible region from $S^n \cap 1^\perp \ni a$ to $S^n \ni u$. Indeed, the minimum over $u$ of $G_u(0)$ is exactly $\frac{1}{e}$, but we think the minimum over $a$ of $G_a(0)$ is given by

$$G_{a_{\text{facet}}}(0) = \sqrt{\frac{n}{n+1}} \left( \frac{n}{n+1} \right)^{n-1}.$$  

But certainly

$$\frac{1}{e} = \min_u G_u(0) \leq \min_a G_a(0) \leq G_{a_{\text{facet}}}(0),$$

and since $G_{a_{\text{facet}}}(0) = \frac{1}{e} (1 + o(1))$, we lost at most a $1 + o(1)$ factor by expanding the feasible region.
$G_u(x)$ is the density of a sum of independent centered exponentials $u_j(Y_j - 1)$, so $G_u$ is a convolution $f_1 \ast f_2 \ast \cdots \ast f_{n+1}$.
$G_u(x)$ is the density of a sum of independent centered exponentials $u_j(Y_j - 1)$, so $G_u$ is a convolution $f_1 * f_2 * \cdots * f_{n+1}$.

Here, $f_j(x)$ is the density of $u_j(Y_j - 1)$. It’s given by $f_j(x) = \frac{1}{|u_j|} f(\frac{x}{u_j} + 1)$ where $f$ is the density of the standard (uncentered) exponential with mean 1:

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
Tool: Fourier analysis

Take the Fourier transform. Convolution becomes pointwise multiplication.
Tool: Fourier analysis

Take the Fourier transform. Convolution becomes pointwise multiplication.

\[ \hat{f}(t) = \frac{1}{1 + it} \]
Tool: Fourier analysis

Take the Fourier transform. Convolution becomes pointwise multiplication.

\[
\hat{f}(t) = \frac{1}{1 + it}
\]

\[
\hat{f}_j(t) = \frac{e^{iujt}}{1 + iujt}
\]
Tool: Fourier analysis

Take the Fourier transform. Convolution becomes pointwise multiplication.

\[ \hat{f}(t) = \frac{1}{1 + it} \]

\[ \hat{f}_j(t) = \frac{e^{iu_j t}}{1 + iu_j t} \]

\[ \hat{G}_u(t) = \prod_{j=1}^{n+1} \hat{f}_j(t) = \prod_{j=1}^{n+1} \frac{e^{iu_j t}}{1 + iu_j t} \]
Tool: Fourier analysis

Take the Fourier transform. Convolution becomes pointwise multiplication.

\[ \hat{f}(t) = \frac{1}{1 + it} \]

\[ \hat{f}_j(t) = \frac{e^{iuj_t}}{1 + iuj_t} \]

\[ \hat{G}_u(t) = \prod_{j=1}^{n+1} \hat{f}_j(t) = \prod_{j=1}^{n+1} \frac{e^{iuj_t}}{1 + iuj_t} \]

Fourier inversion formula, valid if \( u \) has at least two nonzero entries:

\[ G_u(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{G}_u(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \prod_{j=1}^{n+1} \frac{e^{iuj_t}}{1 + iuj_t} \, dt \]
We wanted to show $G_u(0) \geq \frac{1}{e}$, and this is equivalent to

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \prod_{j=1}^{n+1} \frac{e^{iu_j t}}{1 + iu_j t} \, dt \geq \frac{1}{e}.$$
Tool: Fourier analysis

We wanted to show \( G_u(0) \geq \frac{1}{e} \), and this is equivalent to

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \prod_{j=1}^{n+1} \frac{e^{iu_j t}}{1 + iu_j t} \, dt \geq \frac{1}{e}.
\]

Letting \( F_u(t) := \prod_{j=1}^{n+1} \frac{e^{iu_j t}}{1 + iu_j t} \), we just want to show

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} F_u(t) \, dt \geq \frac{1}{e}.
\]
Thus far, all the techniques have been known.
Thus far, all the techniques have been known. The main difficulty now is estimating the highly oscillatory integral
\[ \int_{-\infty}^{+\infty} F_u(t) \, dt. \]
Thus far, all the techniques have been known. The main difficulty now is estimating the highly oscillatory integral
\[ \int_{-\infty}^{+\infty} F_u(t) \, dt. \]
I’ll spare you the pictures from my first attempt. It really wasn’t great.
Tool: moving the contour of integration

New idea: *moving the contour of integration.*
New idea: *moving the contour of integration.*

- Recall from complex analysis that the integral of a meromorphic function doesn’t depend on the path taken (with some caveats).
New idea: *moving the contour of integration*.

- Recall from complex analysis that the integral of a meromorphic function doesn’t depend on the path taken (with some caveats).

- If we want to estimate the integral $\int_{-\infty}^{+\infty} F_u(t) \, dt$, we can change the contour of integration, from the real line, to a special curve $\gamma_u$. 

16 / 27
New idea: **moving the contour of integration**.

- Recall from complex analysis that the integral of a meromorphic function doesn’t depend on the path taken (with some caveats).
- If we want to estimate the integral $\int_{-\infty}^{+\infty} F_u(t) \, dt$, we can change the contour of integration, from the real line, to a special curve $\gamma_u$.
- We will choose $\gamma_u$ to have the property that $F_u$ is always a positive real number along $\gamma_u$. 
Here’s a plot of $F_u(t)$ with $u = (\sqrt{0.42}, \sqrt{0.38}, \sqrt{0.20})$: 
Tool: moving the contour of integration

Here’s a plot of $F_u(t)$ with $u = (\sqrt{0.42}, \sqrt{0.38}, \sqrt{0.20})$:

The color denotes the argument of $F_u(t)$. Red means real. Follow the red color, trace out a curve $\gamma_u$. 
Tool: moving the contour of integration

Black box (basically just the Implicit Function Theorem)

We can always find such a curve \( \gamma_u \), along which \( F_u \) takes positive real values, such that \( \gamma_u \) is \( C^\infty \) and passes through the origin. Moreover, \( \gamma_u \) can be viewed as the graph of an even function \( y_u(x) \) in the \( xy \)-plane (identified with the complex plane in the usual manner).
Tool: moving the contour of integration

Black box (basically just the Implicit Function Theorem)

We can always find such a curve $\gamma_u$, along which $F_u$ takes positive real values, such that $\gamma_u$ is $C^\infty$ and passes through the origin. Moreover, $\gamma_u$ can be viewed as the graph of an even function $y_u(x)$ in the $xy$-plane (identified with the complex plane in the usual manner).

Here’s a plot of $\gamma_u$ with the same $u$ ($u = (\sqrt{0.42}, \sqrt{0.38}, \sqrt{0.20})$):
As long as $u$ has at least two nonzero entries, we have that the integral $\int_{-\infty}^{+\infty} F_u(t) \, dt$ exists and equals $\int_{\gamma_u} F_u(t) \, dt$. Moreover, the integrand $F_u(t)$ is always a positive real number if $t$ is on $\gamma_u$. 

This is the part when we actually move the contour of integration. So we just need to estimate $\int_{\gamma_u} F_u(t) \, dt$. 

Black box (some crude tail bounds)
Black box (some crude tail bounds)

As long as $u$ has at least two nonzero entries, we have that the integral $\int_{-\infty}^{+\infty} F_u(t) \, dt$ exists and equals $\int_{\gamma_u} F_u(t) \, dt$. Moreover, the integrand $F_u(t)$ is always a positive real number if $t$ is on $\gamma_u$. This is the part when we actually move the contour of integration.
Black box (some crude tail bounds)

As long as $u$ has at least two nonzero entries, we have that the integral $\int_{-\infty}^{+\infty} F_u(t) \, dt$ exists and equals $\int_{\gamma_u} F_u(t) \, dt$. Moreover, the integrand $F_u(t)$ is always a positive real number if $t$ is on $\gamma_u$.

This is the part when we actually move the contour of integration.

So we just need to estimate $\int_{\gamma_u} F_u(t) \, dt$. 
Differential equations

Recall that $y_u(x)$ is the function whose graph is $\gamma_u$. 
Recall that $y_u(x)$ is the function whose graph is $\gamma_u$. Defining $\tilde{F}_u(x) := F_u(x + iy_u(x))$, we can compute that

$$\int_{\gamma_u} F_u(t) \, dt = \int_{\mathbb{R}} \tilde{F}_u(x) \, dx$$

since $\tilde{F}_u$ is an even function of $x$. (We changed $dt$ to $dx$.)

So we just need to show

$$\frac{1}{2} \pi \int_{\mathbb{R}} \tilde{F}_u(x) \, dx \geq e.$$

20 / 27
Recall that $y_u(x)$ is the function whose graph is $\gamma_u$. Defining
$\tilde{F}_u(x) := F_u(x + iy_u(x))$, we can compute that
$\int_{\gamma_u} F_u(t) \, dt = \int_{\mathbb{R}} \tilde{F}_u(x) \, dx$ since $\tilde{F}_u$ is an even function of $x$. (We changed $dt$ to $dx$.)
Differential equations

Recall that $y_u(x)$ is the function whose graph is $\gamma_u$. Defining $\tilde{F}_u(x) := F_u(x + iy_u(x))$, we can compute that $\int_{\gamma_u} F_u(t) \, dt = \int_{\mathbb{R}} \tilde{F}_u(x) \, dx$ since $\tilde{F}_u$ is an even function of $x$. (We changed $dt$ to $dx$.)

So we just need to show that $\frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}_u(x) \, dx \geq \frac{1}{e}$. 

\[\frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}_u(x) \, dx \geq \frac{1}{e}.\]
Differential equations

Compute that equality holds if \( u = (1) \in S^0 \); i.e.

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}(1)(x) \, dx = \frac{1}{e}.
\]
Differential equations

Compute that equality holds if $u = (1) \in S^0$; i.e.

$$\frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}(1)(x) \, dx = \frac{1}{e}.$$

If we could show $\tilde{F}_u(x) \geq \tilde{F}(1)(x)$ for each $x$, then we would automatically get

$$\frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}_u(x) \, dx \geq \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}(1)(x) \, dx = \frac{1}{e}$$

as desired.
Differential equations

Compute that equality holds if \( u = (1) \in S^0 \); i.e.

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}_u(x) \, dx = \frac{1}{e}.
\]

If we could show \[ \tilde{F}_u(x) \geq \tilde{F}_u(1)(x) \text{ for each } x \], then we would automatically get

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}_u(x) \, dx \geq \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}_u(1)(x) \, dx = \frac{1}{e}
\]
as desired.

Let’s show the boxed statement.
Compute that equality holds if \( u = (1) \in S^0 \); i.e.

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}_u(x) \, dx = \frac{1}{e}.
\]

If we could show \( \tilde{F}_u(x) \geq \tilde{F}_{(1)}(x) \) for each \( x \), then we would automatically get

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}_u(x) \, dx \geq \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{F}_{(1)}(x) \, dx = \frac{1}{e}
\]

as desired.

Let’s show the boxed statement. From now on, assume \( x > 0 \).
Differential equations

Defining property of \( y_u \)

\[
y_u' = \sum_{j=1}^{n+1} \frac{-y_u + u_j(x^2 + y_u^2)}{x^2 + (\frac{1}{u_j} - y_u)^2} \div \sum_{j=1}^{n+1} \frac{x}{x^2 + (\frac{1}{u_j} - y_u)^2}
\]
Differential equations

Defining property of $y_u$

\[ y'_u = \sum_{j=1}^{n+1} \frac{-y_u + u_j(x^2 + y_u^2)}{x^2 + \left(\frac{1}{u_j} - y_u\right)^2} \bigg/ \sum_{j=1}^{n+1} \frac{x}{x^2 + \left(\frac{1}{u_j} - y_u\right)^2} \]

Corollary

\[ y'_u \leq \frac{-y_u + x^2 + y_u^2}{x} \]
Differential equations

Defining property of $y_u$

$$y_u' = \sum_{j=1}^{n+1} \frac{-y_u + u_j(x^2 + y_u^2)}{x^2 + \left(\frac{1}{u_j} - y_u\right)^2} \left/ \sum_{j=1}^{n+1} \frac{x}{x^2 + \left(\frac{1}{u_j} - y_u\right)^2}\right.$$

Corollary

$$y_u' \leq \frac{-y_u + x^2 + y_u^2}{x}$$

Using this, we can prove

Black box

$$-y(1) \leq y_u \leq y(1) \text{ for all } x > 0.$$

(*)
Compute

\[
\frac{d}{dx} \log \tilde{F}_u(x) = - \left( \sum_{j=1}^{n+1} \frac{x}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2} \right)^2 + \left( \sum_{j=1}^{n+1} \frac{-y_u + u_j(x^2 + y_u^2)}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2} \right)^2
\]
Differential equations

Compute

\[ \frac{d}{dx} \log \tilde{F}_u(x) = - \frac{\left( \sum_{j=1}^{n+1} \frac{x}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2} \right)^2 + \left( \sum_{j=1}^{n+1} \frac{-y_u + u_j (x^2 + y_u^2)}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2} \right)^2}{\sum_{j=1}^{n+1} \frac{x}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2}} \]

Substituting \( u = (1) \) yields

\[ \frac{d}{dx} \log \tilde{F}_{(1)}(x) = - \frac{x^2 + y_{(1)}^2}{x}. \]
Differential equations: Two curious inequalities

Use Cauchy-Schwarz:

\[
\left( \sum_{j=1}^{n+1} \frac{x}{u_j}^2 + \left( \frac{1}{u_j} - y_u \right)^2 \right)^2 \leq \left( \sum_{j=1}^{n+1} \frac{(x/u_j) \cdot u_j}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2} \right)^2 \leq \left( \sum_{j=1}^{n+1} \left( \frac{x}{u_j} \right)^2 \right) \left( \sum_{j=1}^{n+1} u_j^2 \right) \]
Differential equations: Two curious inequalities

Use Cauchy-Schwarz again:

\[
\left( \sum_{j=1}^{n+1} \frac{-y_u + u_j(x^2 + y_u^2)}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2} \right)^2 = \left( \sum_{j=1}^{n+1} \frac{(-y_u/u_j + x^2 + y_u^2)}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2} \cdot u_j \right)^2 \\
\leq \left( \sum_{j=1}^{n+1} \frac{(-y_u/u_j + x^2 + y_u^2)^2}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2} \right) \left( \sum_{j=1}^{n+1} u_j^2 \right) \\
= \sum_{j=1}^{n+1} \frac{(-y_u/u_j + x^2 + y_u^2)^2}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2} \left( \sum_{j=1}^{n+1} u_j^2 \right)^2
\]
Differential equations

Putting it together:

\[
\frac{d}{dx} \log \tilde{F}_u(x) \geq - \frac{\sum_{j=1}^{n+1} \frac{x^2 + y_u^2}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2}}{\sum_{j=1}^{n+1} \frac{x}{x^2 + \left( \frac{1}{u_j} - y_u \right)^2}}
\]

\[
= - \frac{x^2 + y_u^2}{x}
\]

\[
\geq - \frac{x^2 + y_{(1)}^2}{x}
\]

\[
= \frac{d}{dx} \log \tilde{F}_{(1)}(x)
\]

which is sufficient to imply \( \tilde{F}_u(x) \geq \tilde{F}_{(1)}(x) \), as desired.
Thanks