Math 300 Class 26

Monday 11th March 2019

Definition 1 — *Random variable*
Let $(\Omega, \mathbb{P})$ be a discrete probability space and let E be a set. An **E-valued random variable** on $(\Omega, \mathbb{P})$ is a function $X : \Omega \to E$.

The set E is called the **state space** of X.

We think of X as being a 'variable' element of E depending on the outcome of a random process—if the outcome of the random process is ω, then the value of X is $X(\omega)$.

Example 2
Let X be a real-valued random variable on a discrete probability space $(\Omega, \mathbb{P})$. Express the following events as subsets of Ω.

(a) The event that $X = 0$;

(b) The event that $X \in \mathbb{Z}$;

(c) The event that $e^{X^2} > 3X + 4$.

Exercise 3

A biased coin, which shows heads with probability $0 < p < 1$, is flipped n times, where n is some natural number. Let N be the number of heads that show. Describe a probability space $(\Omega, \mathbb{P})$ which models this random process, give an explicit definition of N as a function $\Omega \to E$ (for an appropriate choice of state space E), and compute $\mathbb{P}\{N = k\}$ for each $k \in E$.

Exercise 4

Let $(\Omega, \mathbb{P})$ be a discrete probability space and let X be a random variable on $(\Omega, \mathbb{P})$. Prove that the events $\{X = e\}$ for $e \in E$ are mutually exclusive (i.e. pairwise disjoint).

Definition 5

Let $(\Omega, \mathbb{P})$ be a discrete probability space and let $X : \Omega \to E$ be a random variable. The **probabilty mass function** of X is the function $f_X : E \to [0,1]$ defined by $f_X(e) = \mathbb{P}\{X = e\}$ for all $e \in E$.

Example 6

Let E be a set and let X be an E-valued random variable on a probability space $(\Omega, \mathbb{P})$. Prove that $X_*\mathbb{P}$ is a probability measure on E, where $(X_*\mathbb{P})(A) = \mathbb{P}\{X \in A\}$ for all $A \subseteq E$.

Definition 7

Let $(\Omega, \mathbb{P})$ be a probability space. A family of events $\{A_i \mid i \in I\}$ is **mutually independent** if

$$\mathbb{P}\left(\bigcap_{i\in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i)$$

In particular, events $A_1, A_2, \dots, A_n$ are mutually independent if and only if

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times \cdots \times \mathbb{P}(A_n)$$

Exercise 8

Let $p \in [0,1]$ and suppose that $X_1, X_2, \dots, X_n$ are $\{0,1\}$-valued with

$$f_{X_k}(i) = \begin{cases} 1-p & \text{if } i = 0 \\ p & \text{if } i = 1 \end{cases}$$

for each $k \in [n]$. Assuming the events $\{X_k = i\}$ are mutually independent for $k \in [n]$ and $i \in \{0,1\}$, prove that the $\{0,1,\dots,n\}$-valued random variable $X = X_1 + \cdots + X_n$ satisfies

$$f_X(r) = \binom{n}{r} p^r (1-p)^{n-r}$$

for each $r \in \{0,1,\dots,n\}$.