Math 300 Class 25

Friday 8th March 2019

Definition 1 — *Conditional probability*
Let $(\Omega, \mathbb{P})$ be a probability space and let $B \subseteq \Omega$ be an event with $\mathbb{P}(B) > 0$. The **conditional probability** of an event A given B is defined by

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Intuitively speaking, $\mathbb{P}(A \mid B)$ is the *updated* probability of A upon receiving the knowledge that the event B has occurred.

Exercise 2
Let $(\Omega, \mathbb{P})$ be a probability space and $B \subseteq \Omega$ with $\mathbb{P}(B) > 0$. Prove that $\mathbb{P}(- \mid B)$ is a probability measure on Ω.

Let $\omega \in \Omega$. Then $\mathbb{P}(\{\omega\} \mid B) = \begin{cases} 0 & \text{if } \omega \notin B \quad \because \{\omega\} \cap B = \emptyset \\ \dfrac{\mathbb{P}(\{\omega\})}{\mathbb{P}(B)} & \text{if } \omega \in B \quad \because \{\omega\} \cap B = \{\omega\} \end{cases}$

$$\begin{aligned}
\text{So} \quad \sum_{\omega \in \Omega} \mathbb{P}(\{\omega\} \mid B) &= \sum_{\omega \in B} \mathbb{P}(\{\omega\} \mid B) + \sum_{\omega \notin B} \mathbb{P}(\{\omega\} \mid B) && \text{splitting up } \Sigma \\
&= \sum_{\omega \in B} \frac{\mathbb{P}(\{\omega\})}{\mathbb{P}(B)} + 0 && \text{as noted above} \\
&= \frac{1}{\mathbb{P}(B)} \sum_{\omega \in B} \mathbb{P}(\{\omega\}) && \text{factoring out } \frac{1}{\mathbb{P}(B)} \\
&= \frac{1}{\mathbb{P}(B)} \mathbb{P}(B) && \text{countable additivity} \\
&= 1
\end{aligned}$$

$\Rightarrow \mathbb{P}(- \mid B)$ is a probability measure on Ω.

Theorem 3 — *Bayes's theorem (simple version)*
Let A and B be events in a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Then

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\mathbb{P}(B)}{\mathbb{P}(A)}$$

Proof

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)\,\mathbb{P}(B)}{\mathbb{P}(A)\,\mathbb{P}(B)} = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}$$

□

This form of Bayes's theorem isn't very enlightening, so we will derive a more useful version of it.

Theorem 4 — *Bayes's theorem (slightly more useful version)*
Let A and B be events in a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(A) > 0$ and $0 < \mathbb{P}(B) < 1$. Prove that

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\mathbb{P}(B)}{\mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\mathbb{P}(B^c)}$$

[We have written B^c to denote the event $\Omega \setminus B$.]

Proof

Note that $A = A \cap \Omega = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$
and $(A \cap B) \cap (A \cap B^c) = A \cap (B \cap B^c) = A \cap \emptyset = \emptyset$

so by countable additivity.

$$\begin{aligned}\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c) &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) \\ &= \mathbb{P}[(A \cap B) \cup (A \cap B^c)] \\ &= \mathbb{P}(A)\end{aligned}$$

□

Now substitute into Thm 3.

Exercise 5

A town has 10000 inhabitants, of whom 30 are infected with Disease X. An inhabitant of the town tests positive for Disease X. Given that the test is 99% accurate, what is the probability that the person is infected with Disease X?

<u>Informally</u>: Let $B = \{\text{person is infected}\}$ (so $B^c = \{\text{person is not infected}\}$) & Let $A = \{\text{test is positive}\}$.

By Bayes's thm:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c)}$$

$$= \frac{\frac{99}{100} \cdot \frac{30}{10000}}{\frac{99}{100} \cdot \frac{30}{10000} + \frac{1}{100} \cdot \frac{9970}{10000}}$$

$$= \frac{99 \cdot 30}{99 \cdot 30 + 9970}$$

$$= \frac{2970}{2970 + 9970}$$

$$= \frac{2970}{12940} \approx \underline{\underline{23\%}}$$

Theorem 6 — *Bayes's theorem (even more useful version)*
Let A be an event in a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(A) > 0$, and let $B_1, B_2, \ldots, B_n$ be mutually exclusive events such that $\mathbb{P}(B_i) > 0$ for all $1 \leqslant i \leqslant n$ and such that $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$. Then

$$\mathbb{P}(B_i \mid A) = \frac{\mathbb{P}(A \mid B_i)\mathbb{P}(B_i)}{\mathbb{P}(A \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\mathbb{P}(B_2) + \cdots + \mathbb{P}(A \mid B_n)\mathbb{P}(B_n)}$$

for all $1 \leqslant i \leqslant n$.

Proof. Notice that $A = A \cap \Omega = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$. By countable additivity,

$$\mathbb{P}(A) = \mathbb{P}(A \cap B_1) + \mathbb{P}(A \cap B_2) + \cdots + \mathbb{P}(A \cap B_n)$$

Now observe that $\mathbb{P}(A \cap B_i) = \mathbb{P}(A \mid B_k)\mathbb{P}(B_k)$ for each $k \in [n]$ and substitute into Theorem 3. $\square$

Exercise 7

A small car manufacturer, *Cars N'At*, makes three models of car: the *Allegheny*, the *Monongahela* and the *Ohio*. It made 3000 Alleghenys, 6500 Monongahelas, and 500 Ohios. In a given day, an Allegheny breaks down with probability $\frac{1}{100}$, a Monongahela breaks down with probability $\frac{1}{200}$, and the notoriously unreliable Ohio breaks down with probability $\frac{1}{20}$. An angry driver calls Cars N'At to complain that their car has broken down. Find the probability that the driver was driving an Ohio.

Let $A = \{\text{car broke down}\}$, $B_1 = \{\text{car was an Allegheny}\}$

$B_2 = \{\text{car was a Monongahela}\}$

$B_3 = \{\text{car was an Ohio}\}$

Note: these are mutually exclusive & cover all possibilities ($\Rightarrow B_1 \cup B_2 \cup B_3 = \Omega$)

$$\mathbb{P}(B_3 \mid A) = \frac{\mathbb{P}(A \mid B_3)\mathbb{P}(B_3)}{\mathbb{P}(A \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\mathbb{P}(B_2) + \mathbb{P}(A \mid B_3)\mathbb{P}(B_3)}$$

$$= \frac{\frac{1}{20} \cdot \frac{500}{10000}}{\frac{1}{100} \cdot \frac{3000}{10000} + \frac{1}{200} \cdot \frac{6500}{10000} + \frac{1}{20} \cdot \frac{50}{10000}}$$

$$= \frac{10 \cdot 5}{2 \cdot 30 + 1 \cdot 65 + 10 \cdot 5}$$

$$= \frac{50}{60 + 65 + 50}$$

$$= \frac{10}{12 + 13 + 10}$$

$$= \frac{10}{35}$$

$$= \frac{2}{7} \approx \underline{\underline{29\%}}$$