Math 300 Class 25

Friday 8th March 2019

Definition 1 — *Conditional probability*
Let $(\Omega, \mathbb{P})$ be a probability space and let $B \subseteq \Omega$ be an event with $\mathbb{P}(B) > 0$. The **conditional probability** of an event A given B is defined by

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Intuitively speaking, $\mathbb{P}(A \mid B)$ is the *updated* probability of A upon receiving the knowledge that the event B has occurred.

Exercise 2
Let $(\Omega, \mathbb{P})$ be a probability space and $B \subseteq \Omega$ with $\mathbb{P}(B) > 0$. Prove that $\mathbb{P}(- \mid B)$ is a probability measure on Ω.

Theorem 3 — *Bayes's theorem (simple version)*
Let A and B be events in a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Then

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\mathbb{P}(B)}{\mathbb{P}(A)}$$

Proof

□

This form of Bayes's theorem isn't very enlightening, so we will derive a more useful version of it.

Theorem 4 — *Bayes's theorem (slightly more useful version)*
Let A and B be events in a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(A) > 0$ and $0 < \mathbb{P}(B) < 1$. Prove that

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\mathbb{P}(B)}{\mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^{\mathsf{c}})\mathbb{P}(B^{\mathsf{c}})}$$

[We have written B^{c} to denote the event $\Omega \setminus B$.]

Proof

□

Exercise 5

A town has 10000 inhabitants, of whom 30 are infected with Disease X. An inhabitant of the town tests positive for Disease X. Given that the test is 99% accurate, what is the probability that the person is infected with Disease X?

Theorem 6 — *Bayes's theorem (even more useful version)*

Let A be an event in a probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(A) > 0$, and let $B_1, B_2, \dots, B_n$ be mutually exclusive events such that $\mathbb{P}(B_i) > 0$ for all $1 \leqslant i \leqslant n$ and such that $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$. Then

$$\mathbb{P}(B_i \mid A) = \frac{\mathbb{P}(A \mid B_i)\mathbb{P}(B_i)}{\mathbb{P}(A \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\mathbb{P}(B_2) + \cdots + \mathbb{P}(A \mid B_n)\mathbb{P}(B_n)}$$

for all $1 \leqslant i \leqslant n$.

Proof. Notice that $A = A \cap \Omega = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$. By countable additivity,

$$\mathbb{P}(A) = \mathbb{P}(A \cap B_1) + \mathbb{P}(A \cap B_2) + \cdots + \mathbb{P}(A \cap B_n)$$

Now observe that $\mathbb{P}(A \cap B_i) = \mathbb{P}(A \mid B_k)\mathbb{P}(B_k)$ for each $k \in [n]$ and substitute into Theorem 3. $\square$

Exercise 7

A small car manufacturer, *Cars N'At*, makes three models of car: the *Allegheny*, the *Monongahela* and the *Ohio*. It made 3000 Alleghenys, 6500 Monongahelas, and 500 Ohios. In a given day, an Allegheny breaks down with probability $\frac{1}{100}$, a Monongahela breaks down with probability $\frac{1}{200}$, and the notoriously unreliable Ohio breaks down with probability $\frac{1}{20}$. An angry driver calls Cars N'At to complain that their car has broken down. Find the probability that the driver was driving an Ohio.