Math 300 Class 24

Wednesday 6th March 2019

Definition 1
A **discrete probability space** $(\Omega, \mathbb{P})$ consists of a countable set Ω and a function $\mathbb{P} : \mathscr{P}(\Omega) \to [0,1]$, such that:

(i) $\mathbb{P}(\Omega) = 1$; and

(ii) (**Countable additivity**) For any family $\{A_i \mid i \in I\}$ of pairwise disjoint subsets of Ω indexed by a countable set I, we have

$$\mathbb{P}\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mathbb{P}(A_i)$$

Some terminology:

- The word 'discrete' refers to countability of Ω;
- The set Ω is called the **sample space**, and its elements $\omega \in \Omega$ are called **outcomes**;
- A subset $A \subseteq \Omega$ is called an **event**;
- The function $\mathbb{P} : \mathscr{P}(\Omega) \to [0,1]$ is called a **probability measure** on Ω;
- For each event A, the value $\mathbb{P}(A)$ is called the **probability** of A.

Amazingly, everything we could possibly want to prove about discrete probability spaces can be derived from the two conditions in Definition 1.

Exercise 2
Prove that $\mathbb{P}(\Omega \setminus A) = 1 - \mathbb{P}(A)$ for all events A, and deduce that $\mathbb{P}(\varnothing) = 0$.

Let $A \subseteq \Omega$. Then $A \cap (\Omega \setminus A) = \emptyset$ and $A \cup (\Omega \setminus A) = \Omega$,

so by countable additivity:

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup (\Omega \setminus A)) = \mathbb{P}(A) + \mathbb{P}(\Omega \setminus A)$$

$$\Rightarrow \mathbb{P}(\Omega \setminus A) = 1 - \mathbb{P}(A).$$

Since $\Omega \subseteq \Omega$ and $\emptyset = \Omega \setminus \Omega$, we have

$$\mathbb{P}(\emptyset) = 1 - \mathbb{P}(\Omega) = 1 - 1 = 0. \qquad \square$$

Theorem 3 — *A probability measure is uniquely determined by its values on individual events*
Let Ω be a countable set. Given any subset $\{p_\omega \mid \omega \in \Omega\} \subseteq [0,1]$, if $\sum\limits_{\omega\in\Omega} p_\omega = 1$, then there is a unique probability measure $\mathbb{P}$ on Ω such that $\mathbb{P}(\{\omega\}) = p_\omega$ for all $\omega \in \Omega$.

Proof *(sketch)*
Existence. Define $\mathbb{P}(A) = \sum\limits_{\omega\in A} p_\omega$ for all $A \subseteq \Omega$, and verify conditions (i) and (ii) from Definition 1. Condition (i) is immediate from the assumption that the numbers p_ω sum to 1. Condition (ii) follows from properties of the summation operator $\sum$.

Uniqueness. Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures on Ω such that $\mathbb{P}_1(\{\omega\}) = \mathbb{P}_2(\{\omega\}) = p_\omega$ for each $\omega \in \Omega$. Then by countable additivity ($\star$) we have

$$\mathbb{P}_1(A) = \mathbb{P}_1\left(\bigcup_{\omega\in A}\{\omega\}\right) \overset{\star}{=} \sum_{\omega\in A}\mathbb{P}_1(\{\omega\}) = \sum_{\omega\in A} p_\omega = \sum_{\omega\in A}\mathbb{P}_2(\{\omega\}) \overset{\star}{=} \mathbb{P}_2\left(\bigcup_{\omega\in A}\{\omega\}\right) = \mathbb{P}_2(A)$$

for all $A \subseteq \Omega$, where the steps marked $\star$ follow from countable additivity. So $\mathbb{P}_1 = \mathbb{P}_2$. □

Example 4
Define a probability space that models the roll of a fair die. Which subset of your sample space represents the event that the die roll is prime? What is the probability that this event occurs?

Take $\Omega = [6]$ — the outcome $k \in [6]$ represents the outcome that the die shows 6.

Define $\mathbb{P}$ using Thm 3 by letting $\mathbb{P}(\{k\}) = \frac{1}{6}$ for each $k \in [6]$. [Note $\sum\limits_{k\in[6]} \frac{1}{6} = 6 \cdot \frac{1}{6} = 1$, so this is well-defined.]

The event that the die roll is prime is given by

$$A = \{2, 3, 5\} \subseteq [6]$$

and $$\mathbb{P}(A) = \sum_{k\in A} \mathbb{P}(\{k\}) = 3 \cdot \frac{1}{6} = \frac{1}{2}$$ □

countable additivity

since $|A| = 3$ & $\mathbb{P}(\{k\}) = \frac{1}{6}$ for each $k \in A$.

Example 5

A coin shows heads with probability $p \in [0,1]$, and tails otherwise. Define a probability space that models the random process of flipping the coin until it shows heads; verify that your probability measure is well-defined.

Let $\Omega = \mathbb{N} \cup \{\infty\}$ — $n \in \Omega$ with $n \in \mathbb{N}$ represents the outcome that heads show on the n^{th} flip & tails show on all previous flips, and $\infty \in \Omega$ represents the outcome that heads never shows.

Define $\mathbb{P}$ using Thm 3 by $\begin{cases} \mathbb{P}(\{n\}) = (1-p)^{n-1} p & \text{if } n \in \mathbb{N} \\ \mathbb{P}(\{\infty\}) = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases} \end{cases}$

If $p=0$ then $\sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) = 1 + \sum_{n \in \mathbb{N}} (1-0)^{n-1} \cdot 0 = \underline{\underline{1}}$

If $p>0$ then $\sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) = 0 + \sum_{n \in \mathbb{N}} (1-p)^{n-1} p = p \cdot \frac{1}{1-(1-p)} = \underline{\underline{1}}$.

$\square$

Definition 6

Let $(\Omega, \mathbb{P})$ be a probability space. Events A and B are **independent** if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Exercise 7

"fair die rolled twice"

Let $(\Omega, \mathbb{P})$ be the probability space defined by $\Omega = [6] \times [6]$ and $\mathbb{P}(\{(a,b)\}) = \frac{1}{36}$ for each $(a,b) \in \Omega$. Find events $A, B, C, D \subseteq \Omega$ such that A and B are independent, but C and D are not.

Let $A = \{1\} \times [6] = \{\text{first die roll is } 1\} = C = D$

$B = [6] \times \{1\} = \{\text{second die roll is } 1\}$

Then $\mathbb{P}(A) = \frac{|A|}{36} = \frac{6}{36} = \frac{1}{6} \quad (= \mathbb{P}(C) = \mathbb{P}(D))$

$\mathbb{P}(B) = \frac{|B|}{36} = \frac{6}{36} = \frac{1}{6}$

$A \cap B = \{1\} \times \{1\} = \{(1,1)\} \Rightarrow \mathbb{P}(A \cap B) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = \mathbb{P}(A)\mathbb{P}(B)$

$\Rightarrow$ A & B are independent

but $C \cap D = \{1\} \times [6] \Rightarrow \mathbb{P}(C \cap D) = \frac{1}{6} \neq \frac{1}{6} \cdot \frac{1}{6} = \mathbb{P}(C)\mathbb{P}(D)$

$\Rightarrow$ C & D are not independent

Exercise 8
When is an event independent from itself?

Let $(\Omega, \mathbb{P})$ be a probability space and let $A \subseteq \Omega$.

Then A is independent from A

$\iff \mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A)$

$\iff \mathbb{P}(A) = \mathbb{P}(A)^2$ $\quad \because A \cap A = A$

$\iff \mathbb{P}(A)(1 - \mathbb{P}(A)) = 0$ $\quad$ rearranging

$\iff \underline{\underline{\mathbb{P}(A) = 0}}$ or $\underline{\underline{\mathbb{P}(A) = 1}}$ $\quad$ solving

[Note: $\mathbb{P}(A) = 0$ does not imply $A = \emptyset$
and $\mathbb{P}(A) = 1$ does not imply $A = \Omega$]

Exercise 9
Prove that the relation '*A* and *B* are independent' on the set of all events in a probability space need not be transitive.

We pretty much did this in Ex 7.

Let $(\Omega, \mathbb{P})$, A & B be as in Ex 7.

Then A & B are independent,

& B & A are independent,

but A & A are not independent.

So the relation is not transitive. $\square$