Math 300 Class 24

Wednesday 6th March 2019

Definition 1

A **discrete probability space** $(\Omega, \mathbb{P})$ consists of a countable set Ω and a function $\mathbb{P} : \mathscr{P}(\Omega) \to [0,1]$, such that:

(i) $\mathbb{P}(\Omega) = 1$; and

(ii) (**Countable additivity**) For any family $\{A_i \mid i \in I\}$ of pairwise disjoint subsets of Ω indexed by a countable set I, we have

$$\mathbb{P}\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mathbb{P}(A_i)$$

Some terminology:

- The word 'discrete' refers to countability of Ω;
- The set Ω is called the **sample space**, and its elements $\omega \in \Omega$ are called **outcomes**;
- A subset $A \subseteq \Omega$ is called an **event**;
- The function $\mathbb{P} : \mathscr{P}(\Omega) \to [0,1]$ is called a **probability measure** on Ω;
- For each event A, the value $\mathbb{P}(A)$ is called the **probability** of A.

Amazingly, everything we could possibly want to prove about discrete probability spaces can be derived from the two conditions in Definition 1.

Exercise 2

Prove that $\mathbb{P}(\Omega \setminus A) = 1 - \mathbb{P}(A)$ for all events A, and deduce that $\mathbb{P}(\varnothing) = 0$.

Theorem 3 — *A probability measure is uniquely determined by its values on individual events*
Let Ω be a countable set. Given any subset $\{p_\omega \mid \omega \in \Omega\} \subseteq [0,1]$, if $\sum_{\omega \in \Omega} p_\omega = 1$, then there is a unique probability measure $\mathbb{P}$ on Ω such that $\mathbb{P}(\{\omega\}) = p_\omega$ for all $\omega \in \Omega$.

Proof *(sketch)*
Existence. Define $\mathbb{P}(A) = \sum_{\omega \in A} p_\omega$ for all $A \subseteq \Omega$, and verify conditions (i) and (ii) from Definition 1. Condition (i) is immediate from the assumption that the numbers p_ω sum to 1. Condition (ii) follows from properties of the summation operator $\sum$.

Uniqueness. Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures on Ω such that $\mathbb{P}_1(\{\omega\}) = \mathbb{P}_2(\{\omega\}) = p_\omega$ for each $\omega \in \Omega$. Then by countable additivity ($\star$) we have

$$\mathbb{P}_1(A) = \mathbb{P}_1\left(\bigcup_{\omega \in A} \{\omega\}\right) \stackrel{\star}{=} \sum_{\omega \in A} \mathbb{P}_1(\{\omega\}) = \sum_{\omega \in A} p_\omega = \sum_{\omega \in A} \mathbb{P}_2(\{\omega\}) \stackrel{\star}{=} \mathbb{P}_2\left(\bigcup_{\omega \in A} \{\omega\}\right) = \mathbb{P}_2(A)$$

for all $A \subseteq \Omega$, where the steps marked $\star$ follow from countable additivity. So $\mathbb{P}_1 = \mathbb{P}_2$. □

Example 4
Define a probability space that models the roll of a fair die. Which subset of your sample space represents the event that the die roll is prime? What is the probability that this event occurs?

Example 5

A coin shows heads with probability $p \in [0,1]$, and tails otherwise. Define a probability space that models the random process of flipping the coin until it shows heads; verify that your probability measure is well-defined.

Definition 6

Let $(\Omega, \mathbb{P})$ be a probability space. Events A and B are **independent** if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Exercise 7

Let $(\Omega, \mathbb{P})$ be the probability space defined by $\Omega = [6] \times [6]$ and $\mathbb{P}(\{(a,b)\}) = \frac{1}{36}$ for each $(a,b) \in \Omega$. Find events $A, B, C, D \subseteq \Omega$ such that A and B are independent, but C and D are not.

Exercise 8

When is an event independent from itself?

Exercise 9

Prove that the relation '*A* and *B* are independent' on the set of all events in a probability space need not be transitive.