Math 300 Class 23

Monday 4th March 2019

Definition 1 — *Reflexivity, symmetry and transitivity*
A relation R on a set X is...

- ...**reflexive** if $a \, R \, a$ for all $a \in X$;
- ...**symmetric** if, for all $a, b \in X$, if $a \, R \, b$, then $b \, R \, a$;
- ...**transitive** if, for all $a, b, c \in X$, if $a \, R \, b$ and $b \, R \, c$, then $a \, R \, c$;
- ...an **equivalence relation** if it is reflexive, symmetric and transitive.

Equivalence relations behave in some ways like equality (indeed, equality is reflexive, symmetric and transitive!)—so we will often use symbols like $\sim$ or $\equiv$ or $\approx$, instead of letters like R or S, to denote equivalence relations.

Example 2
Fix $n \in \mathbb{Z}$. Define a relation $\equiv_n$ on $\mathbb{Z}$ by letting $a \equiv_n b$ mean 'n divides $b - a$' for each $a, b \in \mathbb{Z}$. Prove that $\equiv_n$ is an equivalence relation.

Definition 3 — *Equivalence class, quotient*
Let $\sim$ be an equivalence relation on a set X. The **$\sim$-equivalence class** of an element $x \in X$ is the subset $[x]_\sim$ of X defined by
$$[x]_\sim = \{a \in X \mid x \sim a\}$$

If the relation $\sim$ is obvious from context, we may just say 'equivalence class' and write $[x]$, rather than referring to $\sim$ every time.

Example 4

We proved last time that the relation $\sim$ on $\mathbb{R}$ defined by letting $a \sim b$ mean '$a - b \in \mathbb{Q}$' is an equivalence relation. Show that $[0]_\sim = \mathbb{Q}$.

Example 5

Find the equivalence classes of the integers 0, 1 and 2 with respect to the relation $\equiv_3$ on $\mathbb{Z}$, as defined in Example 2.

Definition 6 — *Quotient*

The **quotient** of a set X by an equivalence relation $\sim$ on X is the set $X/{\sim}$ of all $\sim$-equivalence classes of elements of X. That is

$$X/{\sim} = \{\text{equivalence classes of } \sim\} = \{[x]_\sim \mid x \in X\}$$

The quotient of a set by an equivalence relation *identifies* equivalent elements: the relation $\sim$ on X 'becomes' equality on $X/{\sim}$, in the sense that

$$\forall a, b \in X,\ a \sim b \Leftrightarrow [a]_\sim = [b]_\sim$$

Example 7

Describe the set $\mathbb{Z}/{\equiv_3}$.

Example 8

Prove that $|\mathbb{Z}/{\equiv_n}| = n$ for all $n > 0$.

Definition 9
A **partition** of a set X is a collection $\mathscr{A}$ of inhabited subsets of X such that each $x \in X$ is an element of a unique set $U \in \mathscr{A}$.

The next two results prove that partitions and equivalence relations are essentially the same thing: the equivalence classes give a partition of the set, and each partition of X is the quotient of X by a unique equivalence relation.

Example 10
Let X be a set and let $\sim$ be an equivalence relation on X. Prove that $\mathscr{A} = X/\sim$ is a partition of X.

Theorem 11
Let $\mathscr{U}$ be a partition of a set X. There is a unique equivalence relation $\sim$ on X such that $X/\sim = \mathscr{A}$. □