Math 300 Class 22

Friday 1st March 2019

Definition 1 — *Relations*
A **(binary) relation** on a set X is a logical formula $R(x,y)$ with two free variables x,y whose domain of discourse is X. When talking about binary relations, we will write '$x R y$' instead of $R(x,y)$.

Examples of relations:

- 'x divides y' is a relation on $\mathbb{N}$;
- 'x divides y' is a relation on $\mathbb{Z}$;
- '$x - y \in \mathbb{Q}$' is a relation on $\mathbb{R}$;
- Given any set X, the equality relation '$x = y$' is a relation on X;
- The order relations $<, \leqslant, \geqslant, >$ are relations on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$;
- The **empty relation** $\varnothing_X$ on a set X is defined by letting $x \varnothing_X y$ be false for all $x, y \in X$.

Axiom 2 — *Relation extensionality*
Relations R and S on a set X are equal if $\forall x, y \in X,\ x R y \Leftrightarrow x S y$.

Thus relations correspond with subsets of $X \times X$—the subset of $X \times X$ corresponding with a relation R is called the *graph* of R.

Definition 3 — *Graph of a relation*
Let X be a set and let R be a relation on X. The **graph** of R is the subset of $X \times X$ defined by

$$\mathrm{Gr}(R) = \{(x,y) \in X \times X \mid x R y\} \subseteq X \times X$$

We can therefore specify a relation by simply writing down its graph. For example:

- The set $\{(1,1),(1,2),(2,2)\}$ is the graph of the relation $\leqslant$ on $[2]$;
- The set $\{(1,1),(1,2),(2,2)\}$ is also the graph of a relation on $[3]$, but it is not the graph of the relation $\leqslant$ on $[3]$ since $(1,3)$ is not in the set, for example;
- The **diagonal subset** $\{(x,x) \mid x \in X\} \subseteq X \times X$ is the graph of the relation $=$ on a set X;
- The empty set $\varnothing \subseteq X \times X$ is the graph of the empty relation $\varnothing_X$ on a set X;
- The set $X \times X \subseteq X \times X$ is the graph of the **discrete relation** on a set X, which is defined by letting $x R y$ be true for all $x, y \in X$;
- If $f : X \to X$ is a function, its graph $\mathrm{Gr}(f) \subseteq X \times X$ defines a relation R_f on X defined by $x R_f y$ if and only if $f(x) = y$.

Definition 4 — *Reflexivity*
A relation R on a set X is **reflexive** if
$$\forall a \in X, aRa$$

Definition 5 — *Symmetry*
A relation R on a set X is **symmetric** if
$$\forall a,b \in X, (aRb \Rightarrow bRa)$$

Definition 6 — *Antisymmetry*
A relation R on a set X is **antisymmetric** if
$$\forall a,b \in X, [(aRb \wedge bRa) \Rightarrow a=b]$$

Definition 7 — *Transitivity*
A relation R on a set X is **transitive** if
$$\forall a,b,c \in X, [(aRb \wedge bRc) \Rightarrow aRc]$$

Exercise 8
For each of the following relations, determine which of the properties above it satisfies.

(a) R_1 on $\mathbb{Z}$, defined by 'x divides y'

- R_1 is reflexive : Let $n \in \mathbb{Z}$. Then $n = 1 \times n$, so $n R_1 n$.
- R_1 is not symmetric : Let $a=1$ & $b=2$. Then $a R_1 b$ since $2 = 2 \times 1$, but $b \not R_1 a$ since 2 doesn't divide 1.
- R_1 is not antisymmetric : $-1 R_1 1$ since $1 = (-1)(-1)$, and $1 R_1 -1$ since $-1 = (-1) \times 1$, but $-1 \neq 1$.
- R_1 is transitive : We proved this ages ago. Let $a,b,c \in \mathbb{Z}$ & assume $a R_1 b$ & $b R_1 c$. Then $b = qa$ & $c = rb$ for some $q, r \in \mathbb{Z}$. But then $c = rqa$, and $rq \in \mathbb{Z}$, so $a R_1 c$.

(b) R_2 on $\mathbb{N}$, defined by 'x divides y'

- R_2 is reflexive & transitive — the proofs are identical to those for R_1.
- R_2 is not symmetric — again, same as for R_1.
- R_2 is ~~transitive~~ antisymmetric. Let $a, b \in \mathbb{N}$ & assume $a R_2 b$ & $b R_2 a$. Then $b = qa$ and $a = rb$ for some $q, r \in \mathbb{Z}$, so $a = qra$ & $b = qrb$. Hence $a(1-qr) = 0$ & $b(1-qr) = 0$. If $a = 0$ then $b = q \cdot 0 = 0 = a$. If $a \neq 0$ then $1 - qr = 0 \Rightarrow qr = 1 \Rightarrow q = r = 1 \Rightarrow a = 1 \cdot b = b$.

[We may take $q, r \geq 0$ since $a, b \geq 0$.]

In both cases, we have $a = b$.

(c) R_3 on $\mathbb{R}$, defined by '$x-y \in \mathbb{Q}$'

- R_3 is reflexive : Let $a \in \mathbb{R}$. Then $a-a=0=\frac{0}{1} \in \mathbb{Q}$, so $a R_3 a$.
- R_3 is symmetric : Let $a, b \in \mathbb{R}$ & suppose $a R_3 b$. Then $a-b \in \mathbb{Q}$, so $a-b = p/q$ for some $p, q \in \mathbb{Z}$ with $q \neq 0$. But then $b-a = -(a-b) = \frac{-p}{q} \Rightarrow b-a \in \mathbb{Q} \Rightarrow b R_3 a$.
- R_3 is not antisymmetric : $0 R_3 1$ and $1 R_3 0$ since $0-1 = -1/1 \in \mathbb{Q}$ and $1-0 = 1/1 \in \mathbb{Q}$, but $0 \neq 1$.
- R_3 is transitive : Let $a, b, c \in \mathbb{R}$ & assume $a R_3 b$ & $b R_3 c$. Then $a-b \in \mathbb{Q}$ and $b-c \in \mathbb{Q}$, so $\exists p, q, r, s \in \mathbb{Z}$ with $q, s \neq 0$ s.t. $a-b = p/q$ & $b-c = r/s$. But then

$$a-c = (a-b)+(b-c) = p/q + r/s = \frac{ps+qr}{qs} \in \mathbb{Q}$$

$$\Rightarrow a R_3 c.$$

(d) R_4 on [3], defined by $\mathrm{Gr}(R_4) = \{(1,1),(2,3),(3,2),(3,3)\}$.

- R_4 is not reflexive : $2 \not R_4 2$
- R_4 is symmetric : Let $a, b \in [3]$ & assume $a R_4 b$.
  - If $a=1$ then $b=1$, and $1 R_4 1$, so $b R_4 a$.
  - If $a=2$ then $b=3$, and $3 R_4 2$, so $b R_4 a$.
  - If $a=3$ then $b=2$ or 3, but $2 R_4 3$ and $3 R_4 3$, so $b R_4 a$ in both cases.

  In all cases, $b R_4 a$.
- R_4 is not antisymmetric : $2 R_4 3$ & $3 R_4 2$, but $2 \neq 3$
- R_4 is not transitive : $2 R_4 3$ & $3 R_4 2$, but $2 \not R_4 2$.

Exercise 9

Prove that if R is a symmetric, antisymmetric relation on a set X, then it is a subrelation of the equality relation—that is, $\underbrace{\mathrm{Gr}(R) \subseteq \mathrm{Gr}(=)}$.

Equivalently: $\forall a, b \in X,\ a\,R\,b \Rightarrow a = b$.

Let $a, b \in X$ & assume $a\,R\,b$.

Then $b\,R\,a$ since R is symmetric

So $a = b$ since R is antisymmetric. $\square$

Exercise 10

A relation R on a set X is **left-total** if for all $x \in X$, there exists some $y \in X$ such that $x\,R\,y$. Prove that every left-total, symmetric, transitive relation is reflexive.

We need to show $\forall a \in X$, $a\,R\,a$.

So let $a \in X$.

Then $\exists b \in X$ s.t. $a\,R\,b$ $\because$ R is left-total

$\Rightarrow b\,R\,a$ $\because$ R is symmetric

$\Rightarrow a\,R\,a$ $\because$ $a R b$, $b R a$ & R is transitive.

So R is reflexive. $\square$

⋆ ⋆ ⋆ **PSA: There is a pre-class assignment for Class 23 on Monday, March 4th!** ⋆ ⋆ ⋆