Math 300 Class 22

Friday 1st March 2019

Definition 1 — *Relations*
A (**binary**) **relation** on a set X is a logical formula $R(x,y)$ with two free variables x,y whose domain of discourse is X. When talking about binary relations, we will write '$x\,R\,y$' instead of $R(x,y)$.

Examples of relations:

- 'x divides y' is a relation on $\mathbb{N}$;
- 'x divides y' is a relation on $\mathbb{Z}$;
- '$x - y \in \mathbb{Q}$' is a relation on $\mathbb{R}$;
- Given any set X, the equality relation '$x = y$' is a relation on X;
- The order relations $<$, $\leqslant$, $\geqslant$, $>$ are relations on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$;
- The **empty relation** $\varnothing_X$ on a set X is defined by letting $x\,\varnothing_X\,y$ be false for all $x,y \in X$.

Axiom 2 — *Relation extensionality*
Relations R and S on a set X are equal if $\forall x,y \in X,\ x\,R\,y \Leftrightarrow x\,S\,y$.

Thus relations correspond with subsets of $X \times X$—the subset of $X \times X$ corresponding with a relation R is called the *graph* of R.

Definition 3 — *Graph of a relation*
Let X be a set and let R be a relation on X. The **graph** of R is the subset of $X \times X$ defined by

$$\mathrm{Gr}(R) = \{(x,y) \in X \times X \mid x\,R\,y\} \subseteq X \times X$$

We can therefore specify a relation by simply writing down its graph. For example:

- The set $\{(1,1),(1,2),(2,2)\}$ is the graph of the relation $\leqslant$ on $[2]$;
- The set $\{(1,1),(1,2),(2,2)\}$ is also the graph of a relation on $[3]$, but it is not the graph of the relation $\leqslant$ on $[3]$ since $(1,3)$ is not in the set, for example;
- The **diagonal subset** $\{(x,x) \mid x \in X\} \subseteq X \times X$ is the graph of the relation $=$ on a set X;
- The empty set $\varnothing \subseteq X \times X$ is the graph of the empty relation $\varnothing_X$ on a set X;
- The set $X \times X \subseteq X \times X$ is the graph of the **discrete relation** on a set X, which is defined by letting $x\,R\,y$ be true for all $x,y \in X$;
- If $f : X \to X$ is a function, its graph $\mathrm{Gr}(f) \subseteq X \times X$ defines a relation R_f on X defined by $x\,R_f\,y$ if and only if $f(x) = y$.

Definition 4 — *Reflexivity*
A relation R on a set X is **reflexive** if

$$\forall a \in X, a\,R\,a$$

Definition 5 — *Symmetry*
A relation R on a set X is **symmetric** if

$$\forall a, b \in X, (a\,R\,b \Rightarrow b\,R\,a)$$

Definition 6 — *Antisymmetry*
A relation R on a set X is **antisymmetric** if

$$\forall a, b \in X, [(a\,R\,b \wedge b\,R\,a) \;\Rightarrow\; a = b]$$

Definition 7 — *Transitivity*
A relation R on a set X is **transitive** if

$$\forall a, b, c \in X, [(a\,R\,b \wedge b\,R\,c) \;\Rightarrow\; a\,R\,c]$$

Exercise 8
For each of the following relations, determine which of the properties above it satisfies.

(a) R_1 on $\mathbb{Z}$, defined by 'x divides y'

(b) R_2 on $\mathbb{N}$, defined by 'x divides y'

(c) R_3 on $\mathbb{R}$, defined by '$x - y \in \mathbb{Q}$'

(d) R_4 on [3], defined by $\text{Gr}(R_4) = \{(1,1),(2,3),(3,2),(3,3)\}$.

Exercise 9

Prove that if R is a symmetric, antisymmetric relation on a set X, then it is a subrelation of the equality relation—that is, $\mathrm{Gr}(R) \subseteq \mathrm{Gr}(=)$.

Exercise 10

A relation R on a set X is **left-total** if for all $x \in X$, there exists some $y \in X$ such that $x \mathrel{R} y$. Prove that every left-total, symmetric, transitive relation is reflexive.

⋆ ⋆ ⋆ PSA: There is a pre-class assignment for Class 23 on Monday, March 4th! ⋆ ⋆ ⋆