Math 300 Class 20

Monday 25th February 2019

> **Strategy** (Double counting)
> In order to prove that two expressions involving natural numbers are equal, it suffices to define a set X and devise two counting arguments to show that $|X|$ is equal to both expressions.

Example 1
Let $n, k \in \mathbb{N}$. Prove that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

We prove $\binom{n}{k} \cdot k! \cdot (n-k)! = n!$ by double counting.

Define $X = \{$ lists of elements of $[n]$ with each $i \in [n]$ appearing exactly once on the list $\}$

<u>Procedure 1</u> Select a bijection $f : [n] \to X$. This uniquely determines such a list: $f(1), f(2), \dots, f(n)$; each el't of X appears exactly once $\because$ f is a bijection. $\Rightarrow |X| = n!$

<u>Procedure 2</u>

- <u>Step 1</u> Select $U \in \binom{[n]}{k}$. The elements of U will be the first k on the list (in some order). $\longleftarrow \binom{n}{k}$ choices.
- <u>Step 2</u> Put the elements of U in some order in the first k positions in the list. There are $k!$ choices since this amounts to specifying a bijection $[k] \to U$.
- <u>Step 3</u> Put the elements of $[n] \setminus U$ in some order in the last $n-k$ positions in the list. Again there are $(n-k)!$ choices.

By MP, $|X| = \binom{n}{k} k! \cdot (n-k)!$

So $n! = \binom{n}{k} k! (n-k)!$

$$\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \square$$

Example 2

Let $a, b, k \in \mathbb{N}$. Prove that $\sum_{i=0}^{k} \binom{a}{i}\binom{b}{k-i} = \binom{a+b}{k}$.

An animal rescue shelter houses a cats & b dogs. Let X be the set of collections of k animals from the shelter.

<u>Procedure 1</u> Choose k of the $a+b$ animals.
There are $\binom{a+b}{k}$ choices. $\Rightarrow |X| = \binom{a+b}{k}$.

<u>Procedure 2</u> For $0 \le i \le k$, let X_i be the set of collections of k animals in which exactly i cats are chosen. The sets X_i for $0 \le i \le k$ partition X $\because$ at least 0 cats must be chosen & $i \le \underset{\text{chosen}}{\#\text{animals}} = k$.

To count $|X_i|$ for each $0 \le i \le k$

- Step 1 Choose i of the a cats $\longleftarrow \binom{a}{i}$ choices
- Step 2 Choose $\underbrace{k-i}_{\uparrow}$ of the b dogs $\longleftarrow \binom{b}{k-i}$ choices
  $\because$ need exactly k animals

$\Rightarrow |X_i| = \binom{a}{i}\binom{b}{k-i}$ by MP

$\Rightarrow |X| = \sum_{i=0}^{k} \binom{a}{i}\binom{b}{k-i}$ by AP

So $\binom{a+b}{k} = \sum_{i=0}^{k} \binom{a}{i}\binom{b}{k-i}$. $\square$

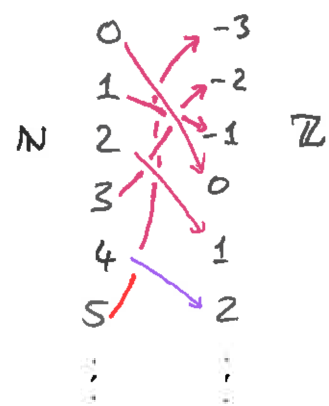
Recall that a set X is *finite* if there is a bijection $[n] \to X$ for some $n \in \mathbb{N}$—this captured the idea that the elements of X can be listed one-by-one in such a way that the list eventually ends. Removing the requirement that the list end reveals the following definition.

Definition 3 — *Countably infinite, countable and uncountable sets*
A set X is **countably infinite** if there is a bijection $\mathbb{N} \to X$. A set is **countable** if it is finite or countably infinite, and is **uncountable** if it is not countable.

Exercise 4
Prove that $\mathbb{N}$ is countably infinite.

$\mathrm{id}_{\mathbb{N}} : \mathbb{N} \longrightarrow \mathbb{N}$ is a bijection. $\square$

Exercise 5
Prove that $\mathbb{Z}$ is countably infinite.



Define $f : \mathbb{N} \longrightarrow \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Then f is a bijection — it has an inverse g defined by

$$g(n) = \begin{cases} 2n & \text{if } n \geqslant 0 \\ -(2n+1) & \text{if } n < 0 \end{cases}$$

Exercise 6
Prove that $\mathscr{P}(\mathbb{N})$ is uncountable.

We show no function $f : \mathbb{N} \longrightarrow \mathscr{P}(\mathbb{N})$ is surjective.

<u>Idea</u> Find $B \in \mathscr{P}(\mathbb{N})$ ("bad element") such that B disagrees with ($\Rightarrow$ is not equal to) each $f(n) \in \mathscr{P}(\mathbb{N})$.

So let $f : \mathbb{N} \longrightarrow \mathscr{P}(\mathbb{N})$ be arbitrary. Define

$$B = \{ n \in \mathbb{N} \mid \underbrace{n \notin f(n)} \}$$

Intuitively: B disagrees with $f(n)$ about whether n is an element, for each $n \in \mathbb{N}$.

Assume $B = f(k)$ for some $k \in \mathbb{N}$.

- If $k \in B$, then $k \notin f(k)$, so $k \notin B$ $\rightsquigarrow$ contradiction
- If $k \notin B$, then $k \in f(k)$, so $k \in B$ $\rightsquigarrow$ contradiction

In both cases we have a contradiction.

So $B \neq f(k)$ for any $k \in \mathbb{N}$

$\Rightarrow f$ is not surjective!

Since no $f : \mathbb{N} \longrightarrow \mathscr{P}(\mathbb{N})$ is surjective,
no $f : \mathbb{N} \longrightarrow \mathscr{P}(\mathbb{N})$ is bijective

$\Rightarrow \mathscr{P}(\mathbb{N})$ is uncountable. $\square$