Math 300 Class 20

Monday 25th February 2019

Strategy (Double counting)
In order to prove that two expressions involving natural numbers are equal, it suffices to define a set X and devise two counting arguments to show that $|X|$ is equal to both expressions.

Example 1

Let $n, k \in \mathbb{N}$ with $k \leqslant n$. Prove that $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Example 2

Let $a, b, k \in \mathbb{N}$. Prove that $\sum_{i=0}^{k} \binom{a}{i}\binom{b}{k-i} = \binom{a+b}{k}$.

Recall that a set X is *finite* if there is a bijection $[n] \to X$ for some $n \in \mathbb{N}$—this captured the idea that the elements of X can be listed one-by-one in such a way that the list eventually ends. Removing the requirement that the list end reveals the following definition.

Definition 3 — *Countably infinite, countable and uncountable sets*
A set X is **countably infinite** if there is a bijection $\mathbb{N} \to X$. A set is **countable** if it is finite or countably infinite, and is **uncountable** if it is not countable.

Exercise 4
Prove that $\mathbb{N}$ is countably infinite.

Exercise 5
Prove that $\mathbb{Z}$ is countably infinite.

Exercise 6

Prove that $\mathscr{P}(\mathbb{N})$ is uncountable.