Math 300 Class 19

Friday 22nd February 2019

> **Strategy** (Addition principle)
> Let X be a finite set. In order to compute $|X|$, it suffices to find a partition $U_1, U_2, \ldots, U_n$ of X; it then follows that $|X| = \sum_{k=1}^{n} |X_i|$.

> **Strategy** (Multiplication principle)
> Let X be a finite set. In order to compute $|X|$, it suffices to find a step-by-step procedure for specifying elements of X, such that:
>
> - Each element is specified by a unique sequence of choices;
> - The number of choices at each step is constant, even if the choices themselves depend on choices made in previous steps.
>
> If there are n steps and m_k possible choices in the k^{th} step, then $|X| = \prod_{k=1}^{n} m_k$.

Example 1
Let $m, n \in \mathbb{N}$. Prove that $|\mathscr{P}([n])| = 2^n$ and $|[n]^{[m]}| = n^m$, where Y^X is the set of functions $X \to Y$.

- Procedure for specifying an element $U \in \mathcal{P}([n])$ (i.e. $U \subseteq [n]$):
    - Step 1 Decide if $1 \in U$ or $1 \notin U$ $\longleftarrow$ 2 choices
    - Step 2 Decide if $2 \in U$ or $2 \notin U$ $\longleftarrow$ 2 choices

      $\vdots$
    - Step n Decide if $n \in U$ or $n \notin U$ $\longleftarrow$ 2 choices

  By MP, $|\mathcal{P}([n])| = \underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$

- Procedure for specifying an element $f \in [n]^{[m]}$ (i.e. $f : [m] \to [n]$):

  m steps; at step k, choose the value of $f(k) \in [n]$

  There are n choices for each $k \in [m]$

  $\Rightarrow$ by MP, $|[n]^{[m]}| = \underbrace{n \times n \times \cdots \times n}_{m \text{ times}} = n^m$. $\square$

Definition 2 — *Factorials (recursive definition)*
Let $n \in \mathbb{N}$. The **factorial** of n is defined by

$$n! = |\{f : [n] \to [n] \mid f \text{ is a bijection}\}|$$

$= \#\text{bijections } X \to Y$ for all X, Y, where $|X| = |Y| = n$

Example 3
Prove that $n! = \prod_{k=1}^{n} k$.

Procedure for specifying a bijection $f : [n] \longrightarrow [n]$:

Step 1 Choose $f(1) \in [n]$ $\leftarrow$ n choices

Step 2 Choose $f(2) \in [n] \setminus \{f(1)\}$ $\leftarrow$ $n-1$ choices

$\vdots$

$(1 \le k \le n)$ Step k Choose $f(k) \in [n] \setminus \{f(1), \ldots, f(k)\}$ $\leftarrow$ $n-k$ choices

$\vdots$

Step n Choose $f(n) \in [n] \setminus \{f(1), \ldots, f(n-1)\}$ $\leftarrow$ 1 choice

By MP, $n! = n \times (n-1) \times \cdots \times 1 = \prod_{k=1}^{n} k$.

[* Note: $n! = |\{f : X \to Y \mid f \text{ is a bijection}\}|$ for all X, Y with $|X| = |Y| = n$. *]

Example 4
Let $n, k \in \mathbb{N}$. Prove that the number of injections $[k] \to [n]$ is $\binom{n}{k} \cdot k!$.

Procedure for specifying an injection $f : [k] \longrightarrow [n]$:

Step 1 Since f will be injective, it will take exactly k values in $[n]$ — so choose $V \in \binom{[n]}{k}$ to be the set of values of f. $\leftarrow$ $\binom{n}{k}$ choices.

Step 2 Choose a bijection $[k] \to V$: this will then determine an injection $[k] \longrightarrow [n]$, with image V. Since $|V| = k$, there are $k!$ choices.

By MP, $(\#\text{injections } [k] \longrightarrow [n]) = \binom{n}{k} \cdot k!$

Strategy (Double counting)
In order to prove that two expressions involving natural numbers are equal, it suffices to define a set X and devise two counting arguments to show that $|X|$ is equal to both expressions.

Example 5
Let $n, k, \ell \in \mathbb{N}$ with $\ell \leqslant n$ and $\ell \leqslant k$. Prove that $\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$.

Let $X = \{(A, B) \mid B \subseteq A \subseteq [n], \quad |A| = k, \quad |B| = \ell\}$

Procedure 1

- Step 1 Choose $A \in \binom{[n]}{k} \longleftarrow \binom{n}{k}$ choices
- Step 2 Choose $B \in \binom{A}{\ell} \longleftarrow \binom{k}{\ell}$ choices $\because |A| = k$

Then $B \subseteq A \subseteq [n]$, $|A| = k$ & $|B| = \ell$, so $|X| = \binom{n}{k}\binom{k}{\ell}$ by MP.

Procedure 2

- Step 1 Choose $B \in \binom{[n]}{\ell} \longleftarrow \binom{n}{\ell}$ choices
- Step 2 Choose $A' \in \binom{[n] \setminus B}{k-\ell} \longleftarrow \binom{n-\ell}{k-\ell}$ choices $\because |[n] \setminus B| = n - \ell$

This determines $(A, B) \in X$ by letting $A = A' \cup B$ — then certainly $B \subseteq A \subseteq [n]$; we have $|B| = \ell$ by construction, and since $A' \subseteq [n] \setminus B$, we have $A' \cap B = \varnothing$, so

$$|A| = |A'| + |B| = (k - \ell) + \ell = k$$

as required.

So by MP, $|X| = \cancel{\binom{n}{k}\binom{k}{\ell}}$ $\binom{n}{\ell}\binom{n-\ell}{k-\ell}$ $\Rightarrow \binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$. $\square$

> **Strategy** (Double counting)
> In order to prove that two expressions involving natural numbers are equal, it suffices to define a set X and devise two counting arguments to show that $|X|$ is equal to both expressions.

Example 5

Let $n,k,\ell \in \mathbb{N}$ with $\ell \leqslant n$ and $\ell \leqslant k$. Prove that $\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$.

Let X be the set of all possible appointments of a k-person committee ($A \subseteq [n]$, $|A| = k$), with an ℓ-person executive subcommittee ($B \subseteq A$, $|B| = \ell$), from a population of size n ($[n]$).

Procedure 1

- Step 1 Choose k people from the population to serve on the committee $\leftarrow \binom{n}{k}$ choices
- Step 2 Choose ℓ people from the committee to serve on the executive subcommittee $\leftarrow \binom{k}{\ell}$ choices

By MP, $|X| = \binom{n}{k}\binom{k}{\ell}$

Procedure 2

- Step 1 Choose ℓ people from the population to serve on the executive subcommittee $\leftarrow \binom{n}{\ell}$ choices
- Step 2 Fill the remaining $k-\ell$ committee positions from the remaining $n-\ell$ people in the population $\leftarrow \binom{n-\ell}{k-\ell}$ choices

By MP, $|X| = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$

By double counting, $\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$.

$\square$

$\#$subsets of $[n]$ of size k ... for $0 \le k \le n$

$\#$subsets of $[n]$

Example 6

Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and that $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$.

- Let $X = \mathcal{P}([n])$

<u>Procedure 1</u> Choose $U \in X$ (i.e. $U \subseteq [n]$) $\leftarrow 2^n$ choices. Done!

So $|X| = 2^n$.

<u>Procedure 2</u> Let $X_k = \{U \subseteq [n] \mid |U| = k\} = \binom{[n]}{k}$, for $0 \le k \le n$.

The sets X_k partition X since if $U \subseteq [n]$ then there is a unique $0 \le k \le n$ s.t. $|U| = k$

By AP, $|X| = \sum_{k=0}^{n} |X_k| = \sum_{k=0}^{n} \binom{n}{k}$. So $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

- Let $X = \{(A, a) \mid A \subseteq [n],\ a \in A\}$.

<u>Procedure 1</u>

Split into cases based on $k = |A|$ — note $0 \le k \le n$.

So define $X_k = \{(A, a) \mid A \in \binom{[n]}{k},\ a \in A\}$. The sets X_k partition X (as above) $\Rightarrow |X| = \sum_{k=0}^{n} |X_k|$, by AP.

For fixed $0 \le k \le n$, we can specify $(A, a) \in X_k$ by:

- <u>Step 1</u> Choose $A \in \binom{[n]}{k} \longleftarrow \binom{n}{k}$ choices
- <u>Step 2</u> Choose $a \in A \longleftarrow k$ choices

$\Rightarrow |X_k| = \binom{n}{k} \cdot k$ by MP

So $|X| = \sum_{k=0}^{n} k\binom{n}{k}$

<u>Procedure 2</u>

- <u>Step 1</u> Choose $a \in [n] \longleftarrow n$ choices
- <u>Step 2</u> Choose $A' \subseteq [n] \setminus \{a\} \longleftarrow 2^{|[n] \setminus \{a\}|} = 2^{n-1}$ choices

Then let $A = A' \cup \{a\}$. This determines $(A, a) \in X$

$\Rightarrow |X| = n \cdot 2^{n-1}$ by MP. So $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$. $\square$

Example 6

Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and that $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$.

- Let X be the set of all possible committees (of any size) from a population of size n.

  <u>Procedure 1</u> Select a subset of the population to serve on the committee $\longleftarrow 2^n$ choices.

  <u>Procedure 2</u> For $0 \le k \le n$, let X_k be the set of committees with exactly k members. These sets partition X, since the size k of any cttee is in the range $0 \le k \le n$ (& the size is uniquely determined)

  $$\Rightarrow |X| = \sum_{k=0}^{n} |X_k| \quad \text{by AP}$$

  To specify an element of X_k for fixed $0 \le k \le n$, just choose k people from the population to serve on the cttee $\longleftarrow \binom{n}{k}$ choices.

  $$\Rightarrow |X_k| = \binom{n}{k} \text{ for all } 0 \le k \le n \quad \Rightarrow \quad 2^n = |X| = \sum_{k=0}^{n} \binom{n}{k}. \quad \square$$

- Now let X be the set of committees from a population of size n, with a distinguished chairperson.

  <u>Procedure 1</u> : <u>Step 1</u> Choose chair from population $\longleftarrow n$ choices

  <u>Step 2</u> Fill remaining positions from the remaining $n-1$ people in the population $\longleftarrow 2^{n-1}$ choices

  $$\Rightarrow |X| = n \cdot 2^{n-1} \text{ by MP.}$$

  <u>Procedure 2</u> : For $0 \le k \le n$, let X_k be the set of possible committees, where there are k people on the committee. As argued above, the sets $\{X_k \mid 0 \le k \le n\}$ partition X.

  To specify an element of X_k: <u>Step 1</u> Choose k people from the pop$^{\underline{n}}$ to serve on cttee $\longleftarrow \binom{n}{k}$ choices

  <u>Step 2</u> Choose chair from the k cttee members $\longleftarrow k$ choices

  $$\Rightarrow |X_k| = \binom{n}{k} \cdot k \text{ by MP, so by AP, } n \cdot 2^{n-1} = |X| = \sum_{k=0}^{n} |X_k| = \sum_{k=0}^{n} k\binom{n}{k}. \quad \square$$

Example 7

Let $n, k \in \mathbb{N}$. Prove that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

We prove $n! = \binom{n}{k} \cdot k! \cdot (n-k)!$ by double counting.

Let $X = \left\{ \begin{array}{l} \text{ordered lists of the numbers } 1, 2, \ldots, n \\ \text{s.t. each number appears exactly once in the list} \end{array} \right\}$

Procedure 1

Choose a bijection $f : [n] \to [n]$. ($n!$ choices)

This determines such a list — for $1 \leq k \leq n$, the value $f(k)$ is exactly the k^{th} number on the list.

$\Rightarrow |X| = n!$

Procedure 2

- Step 1 Choose which elements of $[n]$ will be the first k to appear in the list $\leftarrow \binom{n}{k}$ choices
- Step 2 Choose the order that the first k numbers in the list appear. This amounts to specifying a bijection $[k] \to \{\text{first } k \text{ el'ts on list}\}$ $\leftarrow k!$ choices
- Step 3 Choose the order that the last $n-k$ numbers in the list appear. This amounts to specifying a bijection $[n-k] \to \{\text{last } k \text{ el'ts on list}\}$ $\leftarrow (n-k)!$ choices

This completely determines a list of the numbers $1, 2, \ldots, n$ s.t. each el't appears exactly once.

By MP, $|X| = \binom{n}{k} \cdot k! \cdot (n-k)!$

$\Rightarrow n! = \binom{n}{k} \cdot k! \cdot (n-k)! \Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\square$