Math 300 Class 19

Friday 22nd February 2019

Strategy (Addition principle)
Let X be a finite set. In order to compute $|X|$, it suffices to find a partition $U_1, U_2, \ldots, U_n$ of X; it then follows that $|X| = \sum_{k=1}^{n} |X_i|$.

Strategy (Multiplication principle)
Let X be a finite set. In order to compute $|X|$, it suffices to find a step-by-step procedure for specifying elements of X, such that:

- Each element is specified by a unique sequence of choices;
- The number of choices at each step is constant, even if the choices themselves depend on choices made in previous steps.

If there are n steps and m_k possible choices in the k^{th} step, then $|X| = \prod_{k=1}^{n} m_k$.

Example 1
Let $m, n \in \mathbb{N}$. Prove that $|\mathscr{P}([n])| = 2^n$ and $|[n]^{[m]}| = n^m$, where Y^X is the set of functions $X \to Y$.

Definition 2 — *Factorials (recursive definition)*
Let $n \in \mathbb{N}$. The **factorial** of n is defined by

$$n! = |\{f : [n] \to [n] \mid f \text{ is a bijection}\}|$$

Example 3

Prove that $n! = \prod_{k=1}^{n} k$.

Example 4

Let $n, k \in \mathbb{N}$. Prove that the number of injections $[k] \to [n]$ is $\binom{n}{k} \cdot k!$.

Strategy (Double counting)
In order to prove that two expressions involving natural numbers are equal, it suffices to define a set X and devise two counting arguments to show that $|X|$ is equal to both expressions.

Example 5

Let $n, k, \ell \in \mathbb{N}$ with $\ell \leqslant n$ and $\ell \leqslant k$. Prove that $\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$.

Example 6

Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and that $\sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$.

Example 7

Let $n, k \in \mathbb{N}$. Prove that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Example 8

Let $n \in \mathbb{N}$. Prove that $\displaystyle\sum_{k=1}^{n} \sum_{\ell=0}^{n-k} k \binom{n}{k} \binom{n-k}{\ell} = n \cdot 3^{n-1}$.