Math 300 Class 18

Wednesday 20th February 2019

Theorem 1 — *Some properties of size*

(a) If Y is finite and there is an injection $X \to Y$, then X is finite and $|X| \leqslant |Y|$;

(b) If X is finite and there is a surjection $X \to Y$, then Y is finite and $|X| \geqslant |Y|$;

(c) If X and Y are finite, then $X \times Y$ is finite and $|X \times Y| = |X| \cdot |Y|$;

(d) If X and Y are finite and $X \cap Y = \varnothing$, then $X \cup Y$ is finite and $|X \cup Y| = |X| + |Y|$. □

Definition 2 — *Binomial coefficients (combinatorial definition)*

Let $n, k \in \mathbb{N}$. The set $\binom{[n]}{k}$ is defined by

$$\binom{[n]}{k} = \{U \subseteq [n] \mid |U| = k\}$$

The **binomial coefficient** $\binom{n}{k}$ is defined by $\binom{n}{k} = \left|\binom{[n]}{k}\right|$.

Example 3

Compute $\binom{3}{k}$ for all $k \in \mathbb{N}$.

Useful fact: $\binom{n}{k} = \left|\binom{X}{k}\right|$ for *any* set X with $|X| = n$.

Parts (a) and (b) of Theorem 1 combine to give the following useful proof technique.

Strategy (Bijective proof)
In order to prove that finite sets X and Y have the same size, it suffices to find a bijection $X \to Y$.

Example 4
Prove that $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ with $k \leqslant n$.

Definition 5

A **partition** of a finite set X is a family $U_1, U_2, \ldots, U_n$ of (inhabited†) subsets of X such that:

(i) $\bigcup_{i=1}^{n} U_i = X$; and

(ii) $U_i \cap U_j = \varnothing$ if $i \neq j$ (that is to say that $U_1, \ldots, U_n$ are **pairwise disjoint**).

[†In the current context, we will additionally allow the sets U_i to be empty.]

Theorem 6 — *Addition principle*

Let X be a finite set and $U_1, U_2, \ldots, U_n$ be a partition of X. Then $|X| = \sum_{i=1}^{n} |U_i|$. □

Strategy 7

In order to count the elements of a set X, it suffices to partition X into subsets $U_1, \ldots, U_n$ and add up the sizes of the sets in the partition.

Example 8

Prove that, for all $n, k \in \mathbb{N}$, we have $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

Theorem 9 — *Multiplication principle*
Fix $m, n \in \mathbb{N}$. Let X be a finite set with $|X| = m$, and for each $a \in X$, let Y_a be a finite set with $|Y_a| = n$. Then

$$|\{(a,b) \mid a \in X,\ b \in Y_a\}| = mn$$

The pair (a,b) is called a **dependent pair**, because the set that b belongs to depends on the value of a. This generalises (by induction!) to sets of dependent n-tuples—the precise statement is ugly.

Strategy 10
Given a finite set X, in order to compute $|X|$, it suffices to devise a step-by-step procedure for uniquely specifying an element of X—each step may depend on the last, but

Example 11
Fix $n, k \geqslant 1$. Compute the size of the set $X = \{(A,a) \mid A \subseteq [n],\ |A| = k,\ a \in A\}$ as follows:

(a) Specify $(A,a) \in X$ by first choosing A and then choosing a.

(b) Specify $(A,a) \in X$ by first choosing a and then choosing A.

Strategy (Double counting)
In order to prove that two expressions involving natural numbers are equal, it suffices to define a set X and devise two counting proofs to show that $|X|$ is equal to both expressions.