Math 300 Class 17

Monday 18th February 2019

Definition 1
Given $n \in \mathbb{N}$, the set $[n]$ is defined by $[n] = \{k \in \mathbb{N} \mid 1 \leqslant k \leqslant n\}$.

Definition 2 — *Finite and infinite sets*
A set X is **finite** if there exists a bijection $f : [n] \to X$ for some $n \in \mathbb{N}$; the function f is called an **enumeration** of X. If X is not finite we say it is **infinite**.

Theorem 3 — *Uniqueness of size*
Let X be a finite set and let $f : [m] \to X$ and $g : [n] \to X$ be enumerations of X. Then $m = n$.

The proof of this 'obvious' fact is a surprisingly complicated induction argument—you can read all about it in §3.2 of the book.

Definition 4 — *Size of a finite set*
Let X be a finite set. The **size** of X, written $|X|$, is the unique natural number n for which there exists a bijection $[n] \to X$.

Example 5
Prove that $[n]$ is finite and $|[n]| = n$ for all $n \in \mathbb{N}$.

Define $f : [n] \to [n]$ by $f(k) = k$ for all $k \in [n]$.
(That is, $f = \mathrm{id}_{[n]}$.)

Then f is a bijection since it is its own inverse:
for all $k \in [n]$ we have $f(f(k)) = f(k) = k$.

So $[n]$ is finite and $|[n]| = n$. $\square$

Example 6
Prove that every inhabited finite subset of $\mathbb{N}$ has a greatest element.

We prove by induction on $n \geqslant 1$ that, for all $X \subseteq \mathbb{N}$ with $|X| = n$, X has a greatest element.

(BC) Let $X \subseteq \mathbb{N}$ with $|X| = 1$. Then $X = \{k\}$ for some $k \in \mathbb{N}$, so k is the greatest element of X.

(IS) Fix $n \geqslant 1$ & suppose that, for all $X \subseteq \mathbb{N}$ with $|X| = n$, X has a greatest element.

Let $X \subseteq \mathbb{N}$ with $|X| = n+1$. (WTS X has gt'st el't.)

Let $f : [n+1] \to X$ be an enumeration of X.

Then the function $f^- : [n] \longrightarrow X \setminus \{f(n+1)\}$ is a bijection, where $f^-(k) = f(k)$ for all $k \in [n]$.

$\Rightarrow |X \setminus \{f(n+1)\}| = n$, so $X \setminus \{f(n+1)\}$ has a greatest element m. But then:

- If $f(n+1) > m$, then $f(n+1)$ is the greatest element of X.
- If $f(n+1) < m$, then m is the greatest element of X.

In any case, X has a greatest element. $\square$

Theorem 7
$\mathbb{N}$ is infinite.

Proof
Suppose $\mathbb{N}$ is finite. Then $\mathbb{N}$ is an inhabited finite subset of $\mathbb{N}$, so by Example 6, $\mathbb{N}$ has a greatest element, say n. But then $n+1 \in \mathbb{N}$ and $n+1 > n$, contradicting maximality of n. So $\mathbb{N}$ is infinite. $\square$

Theorem 8 — *Some properties of size*

(a) If Y is finite and there is an injection $X \to Y$, then X is finite and $|X| \leqslant |Y|$;

(b) If X is finite and there is a surjection $X \to Y$, then Y is finite and $|X| \geqslant |Y|$;

(c) If X and Y are finite, then $X \times Y$ is finite and $|X \times Y| = |X| \cdot |Y|$;

(d) If X and Y are finite and $X \cap Y = \varnothing$, then $X \cup Y$ is finite and $|X \cup Y| = |X| + |Y|$. □

Example 9

Prove that if X is a finite set and $U \subseteq X$, then U is finite and $|U| \leqslant |X|$.

Define $i : U \to X$ by $i(x) = x$ for all $x \in U$.

Then i is injective: given $a, b \in U$, we have

$i(a) = i(b) \Rightarrow a = b$

just by definition of i!

So U is finite and $|U| \leq |X|$ by Thm 8 (a).

Example 10

Prove that if X is a finite set and $U \subseteq X$, then $X \setminus U$ is finite and $|X \setminus U| = |X| - |U|$.

Note that $X \setminus U$ and U are subsets of X, so they are finite by Ex 9.

Moreover $(X \setminus U) \cup U = X$: for all $x \in X$, if $x \in U$ then $x \in (X \setminus U) \cup U$, and if $x \notin U$ then $x \in X \setminus U$, so $x \in (X \setminus U) \cup U$ — this proves $\supseteq$. ($\subseteq$ is immediate since they're subsets of X.)

And $(X \setminus U) \cap U = \emptyset$: if $a \in (X \setminus U) \cap U$ then $a \in X \setminus U$, so $a \notin U$, and $a \in U$ ... contradiction!

So by (d) we have $|X| = |(X \setminus U)| + |U| \Rightarrow |X \setminus U| = |X| - |U|$.

□

Strategy (Bijective proof)
In order to prove that finite sets X and Y have the same size, it suffices to find a bijection $X \to Y$.

Example 11
Let X be a finite set. Prove that $|\mathscr{P}(X)| = |\{0,1\}^X|$, where $\{0,1\}^X$ is the set of functions $X \to \{0,1\}$.

Define $F : \mathcal{P}(X) \to \{0,1\}^X$ by $F(U) = \chi_U$, where $\chi_U : X \to \{0,1\}$ is the characteristic function of U:

$$\chi_U(x) = \begin{cases} 0 & \text{if } x \notin U \\ 1 & \text{if } x \in U \end{cases}$$

[Note that $\forall x \in X \ \forall U \subseteq X, \quad x \in U \Leftrightarrow \chi_U(x) = 1$]

- F is injective: Let $U, V \subseteq X$ & assume $\chi_U = \chi_V$.
  Then for all $a \in X$ we have
  $$a \in U \Leftrightarrow \chi_U(a) = 1 \Leftrightarrow \chi_V(a) = 1 \Leftrightarrow a \in V$$
  So $U = V$ by double containment.

- F is surjective: Let $f : X \to \{0,1\}$. We need to find $U \subseteq X$ such that $f = F(U)$ $(= \chi_U)$.

  So define $U = f^{-1}[\{1\}]$.

  Then for all $a \in X$ we have
  $$f(a) = 1 \Leftrightarrow a \in f^{-1}[\{1\}] \Leftrightarrow a \in U \Leftrightarrow \chi_U(a) = 1$$
  Hence also $f(a) = 0 \Leftrightarrow \chi_U(a) = 0$. So $f(a) = \chi_U(a)$ for all $a \in X$, and hence $f = \chi_U = F(U)$, as required.

So F is a bijection, and so $|\mathcal{P}(X)| = |\{0,1\}^X|$. $\square$