Math 300 Class 17

Monday 18th February 2019

Definition 1
Given $n \in \mathbb{N}$, the set $[n]$ is defined by $[n] = \{k \in \mathbb{N} \mid 1 \leqslant k \leqslant n\}$.

Definition 2 — *Finite and infinite sets*
A set X is **finite** if there exists a bijection $f : [n] \to X$ for some $n \in \mathbb{N}$; the function f is called an **enumeration** of X. If X is not finite we say it is **infinite**.

Theorem 3 — *Uniqueness of size*
Let X be a finite set and let $f : [m] \to X$ and $g : [n] \to X$ be enumerations of X. Then $m = n$.

The proof of this 'obvious' fact is a surprisingly complicated induction argument—you can read all about it in §3.2 of the book.

Definition 4 — *Size of a finite set*
Let X be a finite set. The **size** of X, written $|X|$, is the unique natural number n for which there exists a bijection $[n] \to X$.

Example 5
Prove that $[n]$ is finite and $|[n]| = n$ for all $n \in \mathbb{N}$.

Example 6

Prove that every inhabited finite subset of $\mathbb{N}$ has a greatest element.

Theorem 7

$\mathbb{N}$ is infinite.

Proof

Suppose $\mathbb{N}$ is finite. Then $\mathbb{N}$ is an inhabited finite subset of $\mathbb{N}$, so by Example 6, $\mathbb{N}$ has a greatest element, say n. But then $n+1 \in \mathbb{N}$ and $n+1 > n$, contradicting maximality of n. So $\mathbb{N}$ is infinite. $\square$

Theorem 8 — *Some properties of size*

(a) If Y is finite and there is an injection $X \to Y$, then X is finite and $|X| \leqslant |Y|$;

(b) If X is finite and there is a surjection $X \to Y$, then Y is finite and $|X| \geqslant |Y|$;

(c) If X and Y are finite, then $X \times Y$ is finite and $|X \times Y| = |X| \cdot |Y|$;

(d) If X and Y are finite and $X \cap Y = \varnothing$, then $X \cup Y$ is finite and $|X \cup Y| = |X| + |Y|$. □

Example 9

Prove that if X is a finite set and $U \subseteq X$, then U is finite and $|U| \leqslant |X|$.

Example 10

Prove that if X is a finite set and $U \subseteq X$, then $X \setminus U$ is finite and $|X \setminus U| = |X| - |U|$.

Strategy (Bijective proof)
In order to prove that finite sets X and Y have the same size, it suffices to find a bijection $X \to Y$.

Example 11

Let X be a finite set. Prove that $|\mathscr{P}(X)| = |\{0,1\}^X|$, where $\{0,1\}^X$ is the set of functions $X \to \{0,1\}$.