Math 300 Class 15

Monday 11th February 2019

Recall from Friday's class:

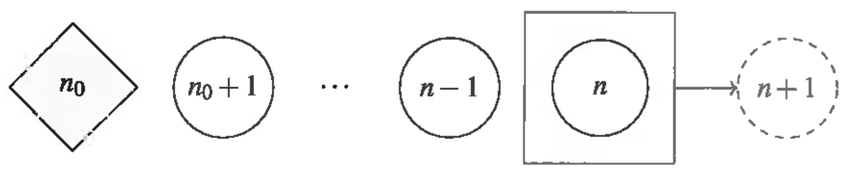
Theorem 1 — *Weak induction principle*
Let $p(n)$ be a logical formula with free variable $n \in \mathbb{N}$ and let $n_0 \in \mathbb{N}$. If

(i) $p(n_0)$ is true; and

(ii) For all $n \geqslant n_0$, if $p(n)$ is true, then $p(n+1)$ is true;

then $p(n)$ is true for all $n \in \mathbb{N}$.

We can illustrate how the weak induction principle works diagramatically as follows.



The shaded diamond represents the base case $p(n_0)$; the square represents the induction hypothesis $p(n)$; and the dashed circle represents the induction goal $p(n+1)$; and the arrow represents the implication we must prove in the induction step. This will help us to make sense of other induction principles.

Theorem 2 — *Strong induction principle*
Let $p(n)$ be a logical formula with free variable $n \in \mathbb{N}$. If

(i) $p(n_0)$ is true; and

(ii) For all $n \geqslant n_0$, if $p(k)$ is true for all $n_0 \leqslant k \leqslant n$, then $p(n+1)$ is true;

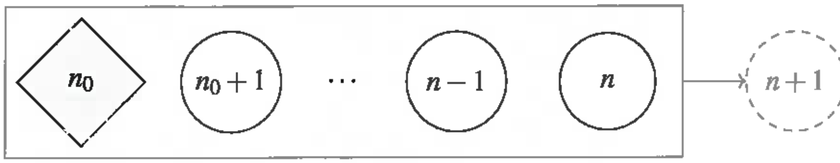
then $p(n)$ is true for all $n \geqslant n_0$.

Proof
The strong induction principle can be proved by weak induction! Define $q(n)$ to mean '$p(k)$ is true for all $n_0 \leqslant k \leqslant n$'. Then

- **(Base case)** $q(n_0)$ is equivalent to $p(n_0)$, which is true by **(i)**.
- **(Induction step)** Fix $n \geqslant n_0$ and assume $q(n)$ is true. Then $p(k)$ is true for all $n_0 \leqslant k \leqslant n$, so that $p(n+1)$ is true by (ii). Hence $p(k)$ is true for all $n_0 \leqslant k \leqslant n+1$, so that $q(n+1)$ is true.

Hence $q(n)$ is true for all $n \geqslant n_0$ by (weak) induction. But then $p(n)$ is also true for all $n \geqslant n_0$, since given $n \geqslant n_0$, if $p(k)$ is true for all $n_0 \leqslant k \leqslant n$, then in particular $p(k)$ is true when $k = n$. □

Strategy (Proof by (strong) induction)
In order to prove a proposition of the form $\forall n \in \mathbb{N}, p(n)$, it suffices to prove that $p(0)$ is true and that, for all $n \in \mathbb{N}$, if $p(k)$ is true for all $k \leqslant n$, then $p(n+1)$ is true.



Observe that the only difference from weak induction is the induction hypothesis.

- **Weak induction step:** Fix $n \geqslant n_0$, $\boxed{\text{assume } p(n) \text{ is true}}$, derive $p(n+1)$;
- **Strong induction step:** Fix $n \geqslant n_0$, $\boxed{\text{assume } p(k) \text{ is true for all } n_0 \leqslant k \leqslant n}$, derive $p(n+1)$.

Example 3
Prove that every natural number greater than or equal to two can be expressed as a product of primes. (We regard a prime number as a product of one prime.)

Base case 2 is prime, so is already a product of primes.

Induction step Fix $n \geq 2$ & suppose that for all $2 \leq k \leq n$, k can be expressed as a product of primes.

We need to show $n+1$ can be expressed as a product of primes. Well:

- If $n+1$ is prime, there is nothing to prove.
- If $n+1$ is prime, then $n+1$ has a proper divisor k with $1 < k < n+1$ ($\Rightarrow 2 \leq k \leq n$). So $n+1 = kl$ for some $l \in \mathbb{Z}$, and then $2 \leq l \leq n$ as well.

By IH, $\exists$ primes $p_1, \ldots, p_r, q_1, \ldots, q_s$ such that
$$k = p_1 \times p_2 \times \cdots \times p_r \quad \& \quad l = q_1 \times \cdots \times q_s.$$

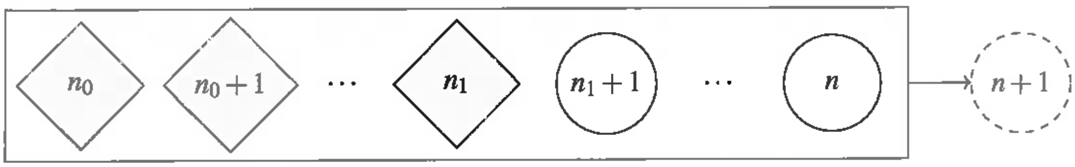
But then $n+1 = kl = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ is a product of primes, as required. $\square$

Theorem 4 — *Strong induction principle (multiple base cases)*
Let $p(n)$ be a logical formula with free variable $n \in \mathbb{N}$ and let $n_0 < n_1 \in \mathbb{N}$. If

(i) $p(n_0), p(n_0+1), \ldots, p(n_1)$ are all true; and

(ii) For all $n \geqslant n_1$, if $p(k)$ is true for all $n_0 \leqslant k \leqslant n$, then $p(n+1)$ is true;

then $p(n)$ is true for all $n \geqslant n_0$.



Example 5
Define a sequence $a_0, a_1, a_2, \ldots$ of natural numbers by

$$a_0 = 0, \quad a_1 = 1, \quad a_n = 3a_{n-1} - 2a_{n-2} \text{ for all } n \geqslant 2$$

Find and prove an expression for a general term a_n in terms of n.

n	0	1	2	3	4	5	$\cdots$	n
a_n	0	1	3	7	15	31	$\cdots$	$2^n - 1$?

Claim $a_n = 2^n - 1$ for all $n \geq 0$.

Proof By induction on n.

- Base cases $a_0 = 0 = 2^0 - 0$ & $a_1 = 1 = 2^1 - 1$ as req'd.
- Induction step Fix $n \geq 1$ & assume $a_k = 2^k - 1$
  $n_1 = 1$
  for all $0 \leq k \leq n$.
  $n_0 = 0$

Then $n+1 \geq 2$, so $a_{n+1} = 3a_n - 2a_{n-1}$ by def. of seq.
$= 3(2^n - 1) - 2(2^{n-1} - 1)$ by IH (since $0 \leq n-1, n \leq n$)
$= 3 \cdot 2^n - 3 - 2^n + 2$
$= 2 \cdot 2^n - 1$
$= 2^{n+1} - 1$ as required.

$\square$

Example 6

The *Fibonacci sequence* begins $0, 1, \ldots$, with subsequent terms generated by adding the previous two terms:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Numbers in this sequence are called *Fibonacci numbers*. Prove that the sum of the squares of two consecutive Fibonacci numbers is a Fibonacci number, and that the difference of squares of Fibonacci numbers spaced two apart in the sequence is a Fibonacci numbers.

Problem setup The Fibonacci sequence is given by

$f_0 = 0, \quad f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Let's look for some patterns:

n	0	1	2	3	4	5	6	7	8	n
f_n	0	1	1	2	3	5	8	13	21	f_n
$f_n^2 + f_{n+1}^2$	1	2	5	13	34		…			f_{2n+1}?
$f_{n+2}^2 - f_n^2$	1	3	8	21	55		…			f_{2n+2}?

Claim For all $n \in \mathbb{N}$, $f_n^2 + f_{n+1}^2 = f_{2n+1}$
and $f_{n+2}^2 - f_n^2 = f_{2n+2}$

Proof By induction on n. ← *Note not sure if strong induction is needed yet so we'll use it just in case.*

Base case $\left\{ \begin{array}{l} f_0^2 + f_1^2 = 0^2 + 1^2 = 1 = f_1 \\ f_2^2 - f_0^2 = 1^2 - 0^2 = 1 = f_2 \end{array} \right.$ as required.

Induction step Fix $n \geq 0$ & assume that

$f_k^2 + f_{k+1}^2 = f_{2k+1}$ and $f_{k+2}^2 - f_k^2 = f_{2k+2}$

for all $0 \leq k \leq n$. We need to prove that

$f_{n+1}^2 + f_{n+2}^2 = f_{2n+3}$

and $f_{n+3}^2 - f_{n+1}^2 = f_{2n+4}$

(continued ……)

By IH we have

$$f_{2n+3} = f_{2n+2} + f_{2n+1} \quad \text{) IH with } k=n.$$
$$= (f_{n+2}^2 - f_n^2) + (f_n^2 + f_{n+1}^2)$$
$$= f_{n+1}^2 + f_{n+2}^2$$

and
$$f_{2n+4} = f_{2n+3} + f_{2n+2} \quad \text{) by what we just showed + IH with } k=n$$
$$= (f_{n+1}^2 + f_{n+2}^2) + (f_{n+2}^2 - f_n^2) \quad \text{) } \because f_{n+2} = f_{n+1} + f_n$$
$$= f_{n+2}^2 + f_{n+1}^2 + (f_{n+1} + f_n)^2 - f_n^2 \quad \text{) expanding}$$
$$= f_{n+2}^2 + 2f_{n+1}^2 + 2f_{n+1}f_n \quad \text{) factorising}$$
$$= f_{n+2}^2 + 2f_{n+1}(f_{n+1} + f_n) \quad \text{) } \because f_{n+2} = f_{n+1} + f_n$$
$$= f_{n+2}^2 + 2f_{n+1}f_{n+2} \quad \text{) factorising}$$
$$= f_{n+2}(f_{n+2} + 2f_{n+1}) \quad \text{) } \because f_{n+3} = f_{n+2} + f_{n+1}$$
$$= (f_{n+3} - f_{n+1})(f_{n+3} + f_{n+1})$$
$$= f_{n+3}^2 - f_{n+1}^2 \quad \text{as required.}$$

$\square$

<u>Note</u> weak induction would have been sufficient since we only used the IH with $k=n$. But if in doubt, do strong induction anyway!