# Math 300 Class 15

Monday 11th February 2019

Recall from Friday's class:

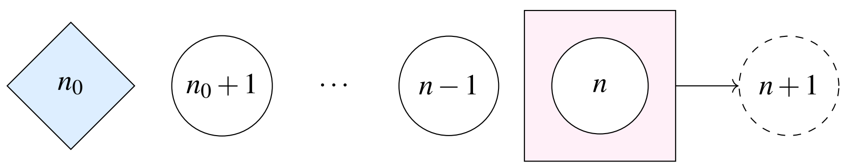
**Theorem 1** — *Weak induction principle*
Let $p(n)$ be a logical formula with free variable $n \in \mathbb{N}$ and let $n_0 \in \mathbb{N}$. If

(i) $p(n_0)$ is true; and

(ii) For all $n \geqslant n_0$, if $p(n)$ is true, then $p(n+1)$ is true;

then $p(n)$ is true for all $n \in \mathbb{N}$.

We can illustrate how the weak induction principle works diagramatically as follows.




The shaded diamond represents the base case $p(n_0)$; the square represents the induction hypothesis $p(n)$; and the dashed circle represents the induction goal $p(n+1)$; and the arrow represents the implication we must prove in the induction step. This will help us to make sense of other induction principles.

**Theorem 2** — *Strong induction principle*
Let $p(n)$ be a logical formula with free variable $n \in \mathbb{N}$. If

(i) $p(n_0)$ is true; and

(ii) For all $n \geqslant n_0$, if $p(k)$ is true for all $n_0 \leqslant k \leqslant n$, then $p(n+1)$ is true;

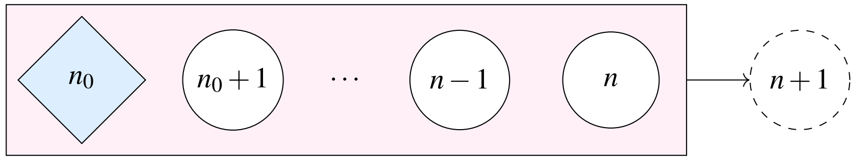
then $p(n)$ is true for all $n \geqslant n_0$.

***Proof***
The strong induction principle can be proved by weak induction! Define $q(n)$ to mean '$p(k)$ is true for all $n_0 \leqslant k \leqslant n$'. Then

- (**Base case**) $q(n_0)$ is equivalent to $p(n_0)$, which is true by (i).
- (**Induction step**) Fix $n \geqslant n_0$ and assume $q(n)$ is true. Then $p(k)$ is true for all $n_0 \leqslant k \leqslant n$, so that $p(n+1)$ is true by (ii). Hence $p(k)$ is true for all $n_0 \leqslant k \leqslant n+1$, so that $q(n+1)$ is true.

Hence $q(n)$ is true for all $n \geqslant n_0$ by (weak) induction. But then $p(n)$ is also true for all $n \geqslant n_0$, since given $n \geqslant n_0$, if $p(k)$ is true for all $n_0 \leqslant k \leqslant n$, then in particular $p(k)$ is true when $k = n$. □

**Strategy** (Proof by (strong) induction)
In order to prove a proposition of the form $\forall n \in \mathbb{N},\ p(n)$, it suffices to prove that $p(0)$ is true and that, for all $n \in \mathbb{N}$, if $p(k)$ is true for all $k \leqslant n$, then $p(n+1)$ is true.




Observe that the only difference from weak induction is the induction hypothesis.

- **Weak induction step:** Fix $n \geqslant n_0$, $\boxed{\text{assume } p(n) \text{ is true}}$, derive $p(n+1)$;
- **Strong induction step:** Fix $n \geqslant n_0$, $\boxed{\text{assume } p(k) \text{ is true for all } n_0 \leqslant k \leqslant n}$, derive $p(n+1)$.

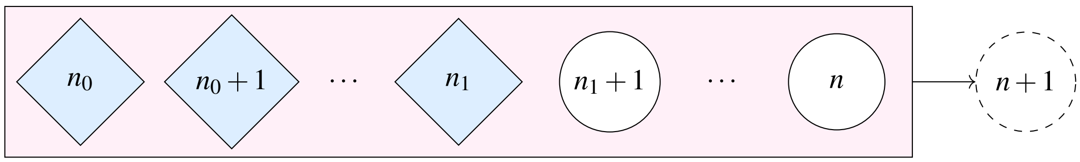
**Example 3**
Prove that every natural number greater than or equal to two can be expressed as a product of primes. (We regard a prime number as a product of one prime.)

**Theorem 4** — *Strong induction principle (multiple base cases)*
Let $p(n)$ be a logical formula with free variable $n \in \mathbb{N}$ and let $n_0 < n_1 \in \mathbb{N}$. If

(i) $p(n_0), p(n_0+1), \ldots, p(n_1)$ are all true; and

(ii) For all $n \geqslant n_1$, if $p(k)$ is true for all $n_0 \leqslant k \leqslant n$, then $p(n+1)$ is true;

then $p(n)$ is true for all $n \geqslant n_0$.




### Example 5

Define a sequence $a_0, a_1, a_2, \ldots$ of natural numbers by

$$a_0 = 0, \quad a_1 = 1, \quad a_n = 3a_{n-1} - 2a_{n-2} \text{ for all } n \geqslant 2$$

Find and prove an expression for a general term $a_n$ in terms of $n$.

### Example 6

The *Fibonacci sequence* begins $0, 1, \ldots$, with subsequent terms generated by adding the previous two terms:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Numbers in this sequence are called *Fibonacci numbers*. Prove that the sum of the squares of two consecutive Fibonacci numbers is a Fibonacci number, and that the difference of squares of Fibonacci numbers spaced two apart in the sequence is a Fibonacci numbers.