

Math 300 Class 12

Monday 4th February 2019

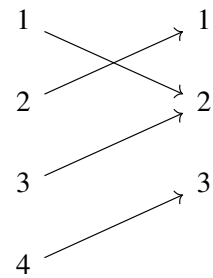
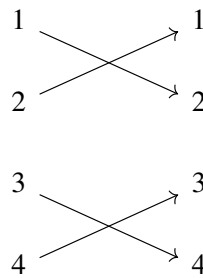
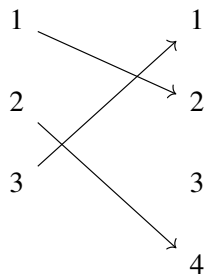
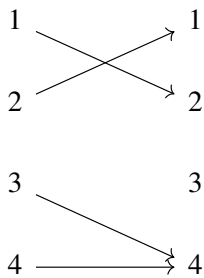
Definition 1

A function $f : X \rightarrow Y$ is...

- ...**injective** if, for all $a, b \in X$, if $f(a) = f(b)$, then $a = b$;
- ...**surjective** if, for all $c \in Y$, there exists $a \in X$ such that $f(a) = c$;
- ...**bijective** if it is injective and surjective.

Example 2

For each of the following diagrams, determine whether the function it represents is: **(B)** bijective, **(I)** injective and not surjective, **(S)** surjective and not injective, or **(N)** neither injective nor surjective.



Example 3

Let $a, b \in \mathbb{R}$ with $a < b$. Find a bijection $(0, 1) \rightarrow (a, b)$.

Example 4

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Prove that if f and g are injective, then $g \circ f$ is injective.
[Note: the same is true with 'injective' replaced by 'surjective' or 'bijective'.]

Definition 5

An **inverse** for a function $f : X \rightarrow Y$ is a function $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

Example 6

Find an inverse for the function you defined in [Example 3](#).

Theorem 7

A function $f : X \rightarrow Y$ is a bijection if and only if it has an inverse.

Proof

(\Rightarrow) Suppose $f : X \rightarrow Y$ is a bijection. Define $g : Y \rightarrow X$ as follows. Given $y \in Y$, there exists $x \in X$ such that $f(x) = y$ since f is surjective. Moreover this element x is unique, since f is injective. So define $g(y) = x$ for the unique $x \in X$ for which $f(x) = y$. Then

- Given $x \in X$, we have $g(f(x))$ is the unique $a \in X$ such that $f(a) = f(x)$, so $g(f(x)) = x$.
- Given $y \in Y$, let $x \in X$ be such that $y = f(x)$. Then we have $f(g(y)) = f(g(f(x))) = f(x) = y$.

So $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$, and so g is an inverse for f .

(\Leftarrow) Assume f has an inverse $g : Y \rightarrow X$. Then

- f is injective. Let $a, b \in X$ and assume that $f(a) = f(b)$. Then $a = g(f(a)) = g(f(b)) = b$.
- f is surjective. Let $c \in Y$ and define $a = g(c)$. Then $f(a) = f(g(c)) = c$.

So f is a bijection. □

Summary of proof strategies for *jections

Strategy (Proving a function is injective)

In order to prove that a function $f : X \rightarrow Y$ is injective, it suffices to fix $a, b \in X$, assume that $f(a) = f(b)$, and then derive $a = b$. ◀

Strategy (Proving a function is surjective)

To prove that a function $f : X \rightarrow Y$ is surjective, it suffices to take an arbitrary element $y \in Y$ and, in terms of y , find an element $x \in X$ such that $f(x) = y$.

In order to find such a value of x , it is often useful to start from the equation $f(x) = y$ and derive some possible values of x . But be careful—in order to complete the proof, it is necessary to verify that the equation $f(x) = y$ is true for the chosen value of x . ◀

Strategy (Proving a function is bijective)

To prove that a function $f : X \rightarrow Y$ is bijective, it suffices to either:

- Prove that f is injective, and that f is surjective; or
- Find an inverse $g : Y \rightarrow X$ for f , and verify that $g(f(x)) = x$ for all $x \in X$ and that $f(g(y)) = y$ for all $y \in Y$. ◀