

Math 300 Class 9

Friday 25th January 2019

Definition 1 — Power set

Let X be a set. The power set of X , written $\mathcal{P}(X)$, is the set of all subsets of X .

Exercise 2

Determine whether or not each of the following statements is true.

(a) $\emptyset \in \{\{\emptyset\}\}$;

False The only element of $\{\{\emptyset\}\}$ is $\{\emptyset\}$,
and $\emptyset \neq \{\emptyset\}$.

(b) $\emptyset \subseteq \{\{\emptyset\}\}$;

True The empty set is a subset of all sets:
 $\forall a, [a \in \emptyset \Rightarrow a \in X]$ is always true
since $a \in \emptyset$ is always false.

(c) $\mathcal{P}(\mathcal{P}(\emptyset)) \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$.

True $\mathcal{P}(\mathcal{P}(\emptyset)) = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
and $\{\emptyset, \{\emptyset\}\} \in \{\emptyset, \underline{\{\emptyset, \{\emptyset\}\}}\}$

(d) $\mathcal{P}(\mathcal{P}(\emptyset)) \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$.

False $\{\emptyset\} \in \mathcal{P}(\mathcal{P}(\emptyset))$

but the only elements of $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$
are \emptyset and $\{\emptyset, \{\emptyset\}\}$, neither of
which is equal to $\{\emptyset\}$.

Definition 3

Let X and Y be sets. The (cartesian) product of X and Y , denoted $X \times Y$, is the set of all ordered pairs (x, y) , where $x \in X$ and $y \in Y$. That is,

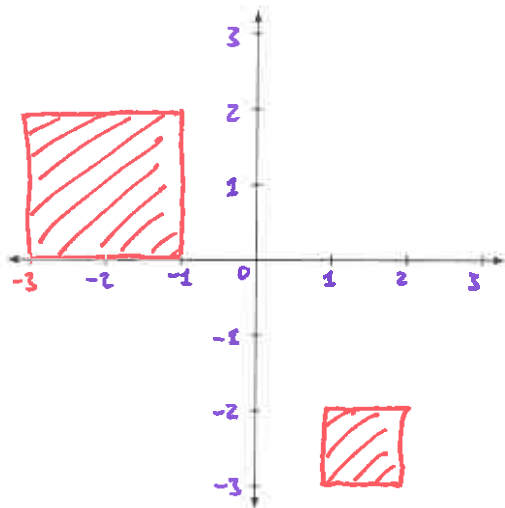
$$X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$$

Ordered pairs satisfy the property that $(x, y) = (a, b)$ if and only if $x = a$ and $y = b$. This is in contrast to sets, which are unordered: for example, $\{0, 1\} = \{1, 0\}$ but $(0, 1) \neq (1, 0)$.

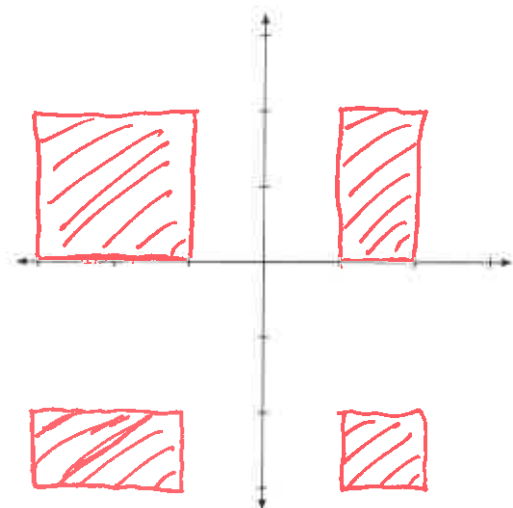
Example 4

On the following pairs axes, sketch the indicated subsets of $\mathbb{R} \times \mathbb{R}$.

$$([-3, -1] \times [0, 2]) \cup ([1, 2] \times [-3, -2])$$



$$([-3, -1] \cup [1, 2]) \times ([0, 2] \cup [-3, -2])$$

**Example 5**

Let A, B, X, Y be sets. Prove that $(A \times X) \cup (B \times Y) \subseteq (A \cup B) \times (X \cup Y)$.

Let $p \in (A \times X) \cup (B \times Y)$. Then $p \in A \times X$ or $p \in B \times Y$.

- If $p \in A \times X$, then $p = (a, x)$ for some $a \in A$ and $x \in X$.
Then $a \in A \cup B$ and $x \in X \cup Y$, so $p \in (A \cup B) \times (X \cup Y)$.
- If $p \in B \times Y$, then $p = (b, y)$ for some $b \in B$ and $y \in Y$.
Then $b \in A \cup B$ and $y \in X \cup Y$, so $p \in (A \cup B) \times (X \cup Y)$.

Hence $(A \times X) \cup (B \times Y) \subseteq (A \cup B) \times (X \cup Y)$.

Indexed unions and intersections

We will often have occasion to take the intersection or union not of just two sets, but of an arbitrary collection of sets (even of infinitely many sets).

Definition 6 — Indexed intersection

The **(indexed) intersection** of a family of sets $\{X_i \mid i \in I\}$ is defined by

$$\bigcap_{i \in I} X_i = \{a \mid \forall i \in I, a \in X_i\}$$

Example 7

Express the set $\bigcap_{n \geq 1} [0, 1 + \frac{1}{n})$ as an interval.

$\bigcap_{n \geq 1} [0, 1 + \frac{1}{n}) = [0, 1]$. The proof of (\supseteq) is fairly easy.*

For (\subseteq) . Let $a \in \bigcap_{n \geq 1} [0, 1 + \frac{1}{n})$. Then $a \in [0, 1 + \frac{1}{n})$ for all $n \geq 1$. So $a \geq 0$. To see $a \leq 1$, note that if

we were to have $a > 1$ then we'd have $a \geq 1 + \frac{1}{n}$ for some sufficiently large n , so $a \notin [0, 1 + \frac{1}{n})$ — contradiction!

So $0 \leq a \leq 1$. So $a \in [0, 1]$.

Definition 8 — Indexed union

The **(indexed) union** of $\{X_i \mid i \in I\}$ is defined by

$$\bigcup_{i \in I} X_i = \{a \mid \exists i \in I, a \in X_i\}$$

Example 9

Express the set $\bigcup_{n \geq 1} (-1 + \frac{1}{n}, 1 - \frac{1}{n})$ as an interval.

$\bigcup_{n \geq 1} (-1 + \frac{1}{n}, 1 - \frac{1}{n}) = (-1, 1)$. Again, one containment is easy* (this time it's (\subseteq))

For (\supseteq) : Let $a \in (-1, 1)$. Then $-1 < a < 1$, so $1 - a > 0$ and $a + 1 > 0$. Let $n \in \mathbb{N}$ be large enough that $\frac{1}{n} < 1 - a$ and $\frac{1}{n} < a + 1$. Then $-1 + \frac{1}{n} < a < 1 - \frac{1}{n}$, so

$a \in (-1 + \frac{1}{n}, 1 - \frac{1}{n})$, and so $a \in \bigcup_{n \geq 1} (-1 + \frac{1}{n}, 1 - \frac{1}{n})$.

Theorem 10 — De Morgan's laws for setsGiven sets A, X, Y and a family of sets $\{X_i \mid i \in I\}$, we have

(a) $A \setminus (X \cup Y) = (A \setminus X) \cap (A \setminus Y);$

(b) $A \setminus (X \cap Y) = (A \setminus X) \cup (A \setminus Y);$

(c) $A \setminus \bigcup_{i \in I} X_i = \bigcap_{i \in I} (A \setminus X_i);$

(d) $A \setminus \bigcap_{i \in I} X_i = \bigcup_{i \in I} (A \setminus X_i).$

Proof of (c)

By double-containment.

(⊆) Let $a \in A \setminus \bigcup_{i \in I} X_i$. Then $a \in A$ and $a \notin \bigcup_{i \in I} X_i$.

So $\neg \exists i \in I, a \in X_i$, so $\forall i \in I, a \notin X_i$. But then $\forall i \in I, a \in A \setminus X_i$, so $a \in \bigcap_{i \in I} (A \setminus X_i)$.

(⊇) Let $a \in \bigcap_{i \in I} (A \setminus X_i)$. Then $\forall i \in I, a \in A$ and $a \notin X_i$.

In particular, $a \in A$; and $\forall i \in I, a \notin X_i$, so $\neg \exists i \in I, a \in X_i$, and so $a \notin \bigcup_{i \in I} X_i$.

Hence $a \in A \setminus \bigcup_{i \in I} X_i$. □

□