Math 300 Class 9

Friday 25th January 2019

Definition 1 — *Power set*
Let X be a set. The **power set** of X, written $\mathscr{P}(X)$, is the set of all subsets of X.

Exercise 2
Determine whether or not each of the following statements is true.

(a) $\varnothing \in \{\{\varnothing\}\}$;

(b) $\varnothing \subseteq \{\{\varnothing\}\}$;

(c) $\mathscr{P}(\mathscr{P}(\varnothing)) \in \{\varnothing, \{\varnothing, \{\varnothing\}\}\}$.

(d) $\mathscr{P}(\mathscr{P}(\varnothing)) \subseteq \{\varnothing, \{\varnothing, \{\varnothing\}\}\}$.

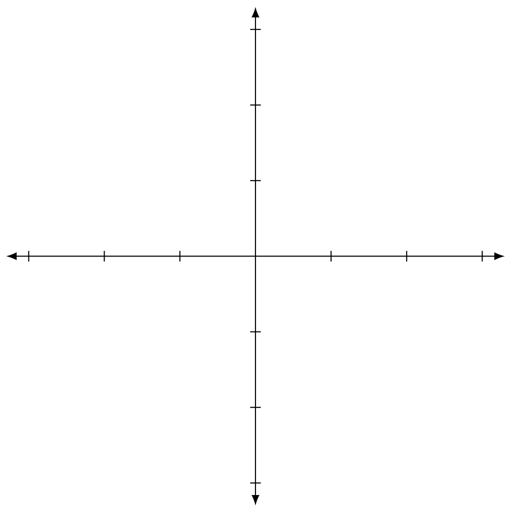
Definition 3
Let X and Y be sets. The (**cartesian**) **product** of X and Y, denoted $X \times Y$, is the set of all **ordered pairs** (x,y), where $x \in X$ and $y \in Y$. That is,

$$X \times Y = \{(x,y) \mid x \in X \wedge y \in Y\}$$

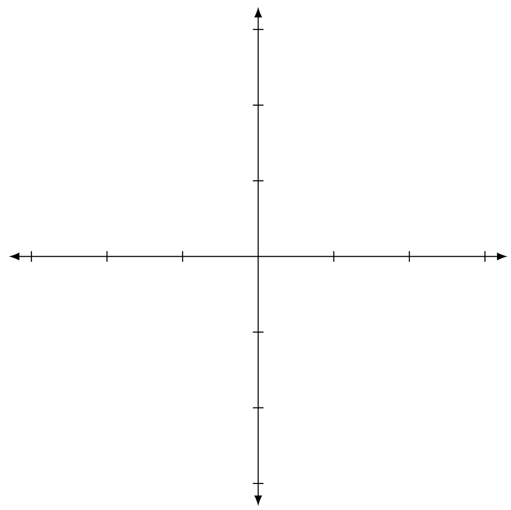
Ordered pairs satisfy the property that $(x,y) = (a,b)$ if and only if $x = a$ and $y = b$. This is in contrast to sets, which are unordered: for example, $\{0,1\} = \{1,0\}$ but $(0,1) \neq (1,0)$.

Example 4
On the following pairs axes, sketch the indicated subsets of $\mathbb{R} \times \mathbb{R}$.

$([-3,-1] \times [0,2]) \cup ([1,2] \times [-3,-2])$



$([-3,-1] \cup [1,2]) \times ([0,2] \cup [-3,-2])$



Example 5
Let A,B,X,Y be sets. Prove that $(A \times X) \cup (B \times Y) \subseteq (A \cup B) \times (X \cup Y)$.

Indexed unions and intersections

We will often have occasion to take the intersection or union not of just two sets, but of an arbitrary collection of sets (even of infinitely many sets).

Definition 6 — *Indexed intersection*
The (**indexed**) **intersection** of a family of sets $\{X_i \mid i \in I\}$ is defined by

$$\bigcap_{i \in I} X_i = \{a \mid \forall i \in I, a \in X_i\}$$

Example 7
Express the set $\bigcap_{n \geqslant 1} [0, 1 + \frac{1}{n})$ as an interval.

Definition 8 — *Indexed union*
The (**indexed**) **union** of $\{X_i \mid i \in I\}$ is defined by

$$\bigcup_{i \in I} X_i = \{a \mid \exists i \in I, a \in X_i\}$$

Example 9
Express the set $\bigcup_{n \geqslant 1} (-1 + \frac{1}{n}, 1 - \frac{1}{n})$ as an interval.

Theorem 10 — *De Morgan's laws for sets*
Given sets A, X, Y and a family of sets $\{X_i \mid i \in I\}$, we have

(a) $A \setminus (X \cup Y) = (A \setminus X) \cap (A \setminus Y)$;

(b) $A \setminus (X \cap Y) = (A \setminus X) \cup (A \setminus Y)$;

(c) $A \setminus \bigcup_{i \in I} X_i = \bigcap_{i \in I} (A \setminus X_i)$;

(d) $A \setminus \bigcap_{i \in I} X_i = \bigcup_{i \in I} (A \setminus X_i)$.

Proof *of (c)*

□