Math 300 Class 8

Tuesday 22nd January 2019

Definition 1 — *Sets and elements*
A **set** is a collection of **elements** from a specified **universe of discourse**. The collection of everything in the universe of discourse is called the **universal set**, denoted by $\mathscr{U}$.

We will avoid referring explicitly to $\mathscr{U}$ whenever possible, but it will always be there in the background. This is convenient because we can abbreviate '$\forall x \in \mathscr{U}, p(x)$' by '$\forall x, p(x)$', and '$\exists x \in \mathscr{U}, p(x)$' by '$\exists x, p(x)$'. Note that under this convention:

- $\forall x \in X, p(x)$ is logically equivalent to $\forall x, (x \in X \Rightarrow p(x))$; and
- $\exists x \in X, p(x)$ is logically equivalent to $\exists x, (x \in X \wedge p(x))$.

Some ways of specifying a set include:

- **Lists** — $\{1,2,3,4,5\}$ or $\{\text{red}, \text{green}, \text{blue}\}$
- **Implied lists** — $\{2,3,5,7,11,13,\ldots\}$ or $\{1,2,4,\ldots,2^n\}$
- **Set-builder notation** — $\{n \in \mathbb{N} \mid n \text{ is prime}\}$ or $\{2^k \mid k \in \mathbb{N} \text{ and } k \leqslant n\}$

Example 2
A *dyadic rational* is a rational number that can be expressed as an integer divided by a power of 2. Express the set of all dyadic rationals using set-builder notation.

$$\left\{ x \in \mathbb{Q} \mid \exists a \in \mathbb{Z}, \exists n \in \mathbb{N}, x = \frac{a}{2^n} \right\} \quad \text{or} \quad \left\{ \frac{a}{2^n} \mid a \in \mathbb{Z}, n \in \mathbb{N} \right\}$$

Definition 3
A set X is **inhabited** if $\exists x, x \in X$ is true; otherwise, it is **empty**.

Example 4
Prove that $\{x \in \mathbb{R} \mid x^2 = 2\}$ is inhabited and $\{x \in \mathbb{Q} \mid x^2 = 2\}$ is empty.

- $\sqrt{2} \in \mathbb{R}$ and $\sqrt{2}^2 = 2$, so $\sqrt{2} \in \{x \in \mathbb{R} \mid x^2 = 2\}$. Hence $\{x \in \mathbb{R} \mid x^2 = 2\}$ is inhabited.
- Suppose $\{x \in \mathbb{Q} \mid x^2 = 2\}$ is inhabited, and let x be an element. Then $x = \pm\sqrt{2}$, both of which are irrational, contradicting the fact that $x \in \mathbb{Q}$. So $\{x \in \mathbb{Q} \mid x^2 = 2\}$ is empty.

It turns out that there is only one empty set, which is denoted by $\varnothing$.

Subsets and set equality

Definition 5
Let X be a set. A **subset** of X is a set U such that $\forall a, (a \in U \Rightarrow a \in X)$. We write $U \subseteq X$ to denote the assertion that U is a subset of X.

Example 6
Prove that $\mathbb{Z} \subseteq \mathbb{Q}$ and $\mathbb{Q} \not\subseteq \mathbb{Z}$.

- Let $n \in \mathbb{Z}$. Then $n = \frac{n}{1} \Rightarrow n \in \mathbb{Q}$. So $\mathbb{Z} \subseteq \mathbb{Q}$.
- The negation of $\forall a, (a \in U \Rightarrow a \in X)$ is $\exists a, a \in U \wedge a \notin X$. So define $a = \frac{1}{2}$. Then $a \in \mathbb{Q}$ (evidently) but $a \notin \mathbb{Z}$, so $\mathbb{Q} \not\subseteq \mathbb{Z}$.

Axiom 7
Two sets X and Y are equal if and only if $\forall a, (a \in X \Leftrightarrow a \in Y)$.

Strategy (Proof of set equality by double containment)
In order to prove $X = Y$, it suffices to prove that $X \subseteq Y$ and $Y \subseteq X$. ◁

Example 8
Prove that $\{x \in \mathbb{R} \mid x^2 < x\} = (0,1)$.

- ($\subseteq$) Let $a \in \{x \in \mathbb{R} \mid x^2 < x\}$. Then $a \in \mathbb{R}$, and $a^2 < a \Rightarrow a - a^2 = a(1-a) > 0$. So
  either $a > 0$ and $1 - a > 0 \Rightarrow 0 < a < 1 \Rightarrow a \in (0,1)$
  or $a < 0$ and $1 - a < 0 \Rightarrow a > 1$ (contradicting $a < 0$).
  So we must have $a \in (0,1)$.
- ($\supseteq$) Let $a \in (0,1)$. Then $0 < a < 1 \overset{(\times a)}{\Rightarrow} 0 < a^2 < a$ and so $a \in \{x \in \mathbb{R} \mid x^2 < x\}$, as required.

So $\{x \in \mathbb{R} \mid x^2 < x\} = (0,1)$ by double containment.

Definition 9 — *Intervals of the real line*
Let $a, b \in \mathbb{R}$. The **open interval** (a,b), the **closed interval** $[a,b]$, and the **half-open intervals** $[a,b)$ and $(a,b]$ from a to b are the subsets of $\mathbb{R}$ defined by

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\} \qquad (a,b] = \{x \in \mathbb{R} \mid a < x \leqslant b\}$$
$$[a,b) = \{x \in \mathbb{R} \mid a \leqslant x < b\} \qquad [a,b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$$

We further define the **unbounded intervals** $(-\infty,a)$, $(-\infty,a]$, $[a,\infty)$ and (a,∞) by

$$(-\infty,a) = \{x \in \mathbb{R} \mid x < a\} \qquad (a,\infty) = \{x \in \mathbb{R} \mid x > a\}$$
$$(-\infty,a] = \{x \in \mathbb{R} \mid x \leqslant a\} \qquad [a,\infty) = \{x \in \mathbb{R} \mid x \geqslant a\}$$

Example 10
For each of the following illustrated intervals of the real line, label it according to the interval that it represents. A filled circle • indicates that an end-point is included in the interval, whereas a hollow circle ∘ indicates that an end-point is not included in the interval.

$(-2,5)$


$[-2,5)$


$[-2,5]$


$(-\infty,5)$


$[2,\infty)$


Definition 11 — *Intersection, union and relative complement*
Let X and Y be sets.

- The **intersection** of X and Y is defined by $X \cap Y = \{a \mid a \in X \wedge a \in Y\}$.
- The **union** of X and Y is defined by $X \cup Y = \{a \mid a \in X \vee a \in Y\}$.
- The **relative complement** of X in Y is defined by $Y \setminus X = \{a \mid a \in Y \wedge a \notin X\}$.

Example 12

Find expressions for each of the following sets as intervals of the real line:

(a) $[-2,5)\cap(4,7]$ $= (4,5)$

($\subseteq$) Let $x \in [-2,5) \cap (4,7]$. Then $x \in [-2,5)$, so $x < 5$, and $x \in (4,7]$, so $x > 4$. Hence $x \in (4,5)$.

($\supseteq$) Let $x \in (4,5)$. Then $x > 4$ and $x < 5 \leq 7$, so $x \in (4,7]$. And $x < 5$ and $x > 4 \geqslant -2$, so $x \in [-2, 4)$.

Hence $x \in [-2,5) \cap (4,7]$.

(b) $[-2,5)\cup(4,7]$ $= [-2,7]$

($\subseteq$) Let $x \in [-2,5) \cup (4,7]$. Then $x \in [-2,5)$ or $x \in (4,7]$. Now

- If $x \in [-2,5)$, then $-2 \leq x < 5 \leq 7 \Rightarrow x \in [-2,7]$
- If $x \in (4,7]$, then $-2 \leq 4 < x \leq 7 \Rightarrow x \in [-2,7]$

So $x \in [-2,7]$ in both cases.

($\supseteq$) Let $x \in [-2,7]$. Then $-2 \leq x \leq 7$. Now

- If $x < 5$, then $x \in [-2,5)$ since $x \geqslant -2$
- If $x \geqslant 5$, then $x > 4$, so $x \in (4,7]$ since $x \leq 7$.

In both cases we have $x \in [-2,5) \cup (4,7]$.

(c) $[-2,5)\setminus(4,7]$ $= [-2,4]$.

($\subseteq$) Let $x \in [-2,5) \setminus (4,7]$. Then $x \in [-2,5)$ and $x \notin (4,7]$. Now $x \geqslant -2$ since $x \in [-2,5)$. If we had $x > 4$, then we'd have $4 < x < 5 \leq 7 \Rightarrow x \in (4,7]$, contradicting the fact that $x \notin (4,7]$. So $x \leq 4$, and hence $x \in [-2,4]$.

($\supseteq$) Let $x \in [-2,4]$. Then

- $-2 \leq x \leq 4 < 5$, so $x \in [-2,5)$; and
- $x \leq 4$, so $x \notin (4,7]$.

Hence $x \in [-2,5) \setminus (4,7]$.