Math 300 Class 8

Tuesday 22nd January 2019

Definition 1 — *Sets and elements*
A **set** is a collection of **elements** from a specified **universe of discourse**. The collection of everything in the universe of discourse is called the **universal set**, denoted by $\mathscr{U}$.

We will avoid referring explicitly to $\mathscr{U}$ whenever possible, but it will always be there in the background. This is convenient because we can abbreviate '$\forall x \in \mathscr{U}, p(x)$' by '$\forall x, p(x)$', and '$\exists x \in \mathscr{U}, p(x)$' by '$\exists x, p(x)$'. Note that under this convention:

- $\forall x \in X, p(x)$ is logically equivalent to $\forall x, (x \in X \Rightarrow p(x))$; and
- $\exists x \in X, p(x)$ is logically equivalent to $\exists x, (x \in X \wedge p(x))$.

Some ways of specifying a set include:

- **Lists** — $\{1,2,3,4,5\}$ or $\{\text{red}, \text{green}, \text{blue}\}$
- **Implied lists** — $\{2,3,5,7,11,13,\dots\}$ or $\{1,2,4,\dots,2^n\}$
- **Set-builder notation** — $\{n \in \mathbb{N} \mid n \text{ is prime}\}$ or $\{2^k \mid k \in \mathbb{N} \text{ and } k \leqslant n\}$

Example 2
A *dyadic rational* is a rational number that can be expressed as an integer divided by a power of 2. Express the set of all dyadic rationals using set-builder notation.

Definition 3
A set X is **inhabited** if $\exists x, x \in X$ is true; otherwise, it is **empty**.

Example 4
Prove that $\{x \in \mathbb{R} \mid x^2 = 2\}$ is inhabited and $\{x \in \mathbb{Q} \mid x^2 = 2\}$ is empty.

It turns out that there is only one empty set, which is denoted by $\varnothing$.

Subsets and set equality

Definition 5
Let X be a set. A **subset** of X is a set U such that $\forall a, (a \in U \Rightarrow a \in X)$. We write $U \subseteq X$ to denote the assertion that U is a subset of X.

Example 6
Prove that $\mathbb{Z} \subseteq \mathbb{Q}$ and $\mathbb{Q} \not\subseteq \mathbb{Z}$.

Axiom 7
Two sets X and Y are equal if and only if $\forall a, (a \in X \Leftrightarrow a \in Y)$.

Strategy (Proof of set equality by double containment)
In order to prove $X = Y$, it suffices to prove that $X \subseteq Y$ and $Y \subseteq X$. $\triangleleft$

Example 8
Prove that $\{x \in \mathbb{R} \mid x^2 < x\} = (0,1)$, where $(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

Definition 9 — *Intervals of the real line*
Let $a, b \in \mathbb{R}$. The **open interval** (a, b), the **closed interval** $[a, b]$, and the **half-open intervals** $[a, b)$ and $(a, b]$ from a to b are the subsets of $\mathbb{R}$ defined by

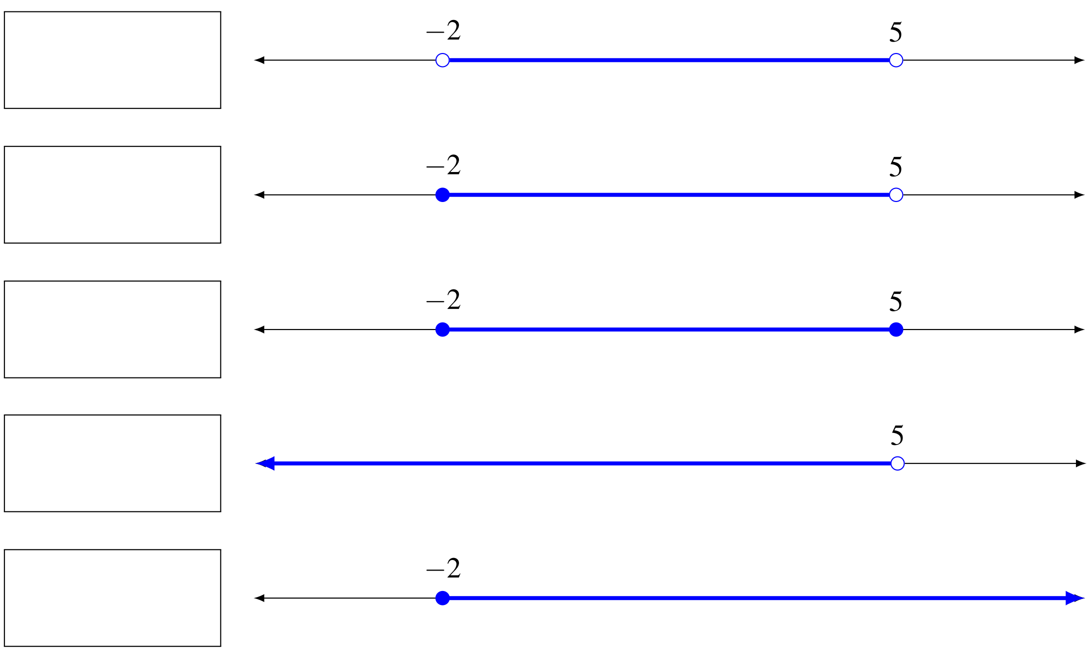
$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\} \qquad (a,b] = \{x \in \mathbb{R} \mid a < x \leqslant b\}$$
$$[a,b) = \{x \in \mathbb{R} \mid a \leqslant x < b\} \qquad [a,b] = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$$

We further define the **unbounded intervals** $(-\infty, a)$, $(-\infty, a]$, $[a, \infty)$ and (a, ∞) by

$$(-\infty,a) = \{x \in \mathbb{R} \mid x < a\} \qquad (a,\infty) = \{x \in \mathbb{R} \mid x > a\}$$
$$(-\infty,a] = \{x \in \mathbb{R} \mid x \leqslant a\} \qquad [a,\infty) = \{x \in \mathbb{R} \mid x \geqslant a\}$$

Example 10

For each of the following illustrated intervals of the real line, label it according to the interval that it represents. A filled circle • indicates that an end-point is included in the interval, whereas a hollow circle ◦ indicates that an end-point is not included in the interval.



Definition 11 — *Intersection, union and relative complement*
Let X and Y be sets.

- The **intersection** of X and Y is defined by $X \cap Y = \{a \mid a \in X \wedge a \in Y\}$.
- The **union** of X and Y is defined by $X \cup Y = \{a \mid a \in X \vee a \in Y\}$.
- The **relative complement** of X in Y is defined by $Y \setminus X = \{a \mid a \in Y \wedge a \notin X\}$.

Example 12

Find expressions for each of the following sets as intervals of the real line:

(a) $[-2,5) \cap (4,7]$

(b) $[-2,5) \cup (4,7]$

(c) $[-2,5) \setminus (4,7]$