Math 300 Class 7

Friday 18th January 2019

So far the only tool we have at our disposal for proving that a proposition is *false* is to assume that it is true and derive a contradiction. In this class we'll use logical equivalence to derive other means of proving when something is false.

Definition 1 — *Maximally negated logical formulae*
A logical formula is **maximally negated** if the only instances of the negation operator $\neg$ appear immediately before a predicate (other proposition involving no logical operators or quantifiers).

Example 2
Identify which of the following logical formulae are maximally negated.

$$[p \wedge (q \Rightarrow (\neg r))] \Leftrightarrow (s \wedge (\neg t)) \qquad\qquad (\neg\neg q) \Rightarrow q$$

$$(\neg p(x)) \Rightarrow \forall y \in X, \neg(r(x,y) \wedge s(x,y)) \qquad\qquad (\neg p(x)) \Rightarrow \forall y \in X, (\neg r(x,y)) \vee (\neg s(x,y))$$

$$\forall x \in \mathbb{R}, [x > 1 \Rightarrow (\exists y \in \mathbb{R}, [x < y \wedge \neg(x^2 \leqslant y)])] \qquad\qquad \neg\exists x \in \mathbb{R}, [x > 1 \wedge (\forall y \in \mathbb{R}, [x < y \Rightarrow x^2 \leqslant y])]$$

Theorem 3
Every logical formula (built using only the logical operators and quantifiers we have seen so far) is logically equivalent to a maximally negated logical formula. □

The precise proof of Theorem 3 is not yet in our reach, but we can derive an algorithm for maximally negating a logical formula by working out how to maximally negate each logical operator and quantifier.

Theorem 4 — *Law of double negation*
Let p be a propositional variable. Then $p \equiv \neg\neg p$.

Proof

□

Theorem 5 — *de Morgan's laws for logical operators*
Let p and q be logical formulae. Then:

(a) $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$; and

(b) $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$.

Proof

□

Theorem 6
Let p and q be logical formulae. Then $\neg(p \Rightarrow q) \equiv p \wedge (\neg q)$.

Proof

□

Theorem 7 — *de Morgan's laws for quantifiers*
let $p(x)$ be a logical formula with free variable x ranging over a set X. Then:

(a) $\neg\forall x \in X,\ p(x) \equiv \exists x \in X,\ \neg p(x)$; and

(b) $\neg\exists x \in X,\ p(x) \equiv \forall x \in X,\ \neg p(x)$.

Proof *of (b)*

□

Part (a) of Theorem 7 is so important that the proof strategy it suggests has a name.

Strategy (Proof by counterexample)
To prove that a proposition of the form $\forall x \in X,\ p(x)$ is false, it suffices to find a single element $a \in X$ such that $p(a)$ is false. The element a is called a **counterexample** to the proposition. ◁

Piecing this all together, we obtain the following, which summarises everything we just proved:

Negation outside		Negation inside	Proof
$\neg(p \wedge q)$	$\equiv$	$(\neg p) \vee (\neg q)$	Theorem 5(a)
$\neg(p \vee q)$	$\equiv$	$(\neg p) \wedge (\neg q)$	Theorem 5(b)
$\neg(p \Rightarrow q)$	$\equiv$	$p \wedge (\neg q)$	Theorem 6
$\neg(\neg p)$	$\equiv$	p	Theorem 4
$\neg \forall x \in X,\, p(x)$	$\equiv$	$\exists x \in X,\, \neg p(x)$	Theorem 7(a)
$\neg \exists x \in X,\, p(x)$	$\equiv$	$\forall x \in X,\, \neg p(x)$	Theorem 7(b)

We can use these equivalences to maximally negate logical formulae by iteratively pushing the negation operator inside the logical formula.

Example 8

Find a maximally negated propositional formula that is logically equivalent to $\neg(p \Leftrightarrow q)$. [It might help you to recall that $p \Leftrightarrow q$ is defined to mean $(p \Rightarrow q) \wedge (q \Rightarrow p)$.]

What strategy does this equivalecne suggest for proving that a proposition of the form $p \Leftrightarrow q$ is false?

Example 9

Maximally negate the following logical formula, then prove that it is true or prove that it is false.

$$\exists x \in \mathbb{R}, [x > 1 \wedge (\forall y \in \mathbb{R}, [x < y \Rightarrow x^2 \leqslant y])]$$