Math 300 Class 4

Friday 11th January 2019

Recall from your pre-class reading:

Definition 1
A **predicate** is a symbol p together with a specified list of **free variables** $x_1, x_2, \ldots, x_n$ and, for each free variable x_i, a specification of a **domain of discourse** of x_i. We will typically write $p(x_1, x_2, \ldots, x_n)$ in order to make the variables explicit.

Definition 2
A **logical formula** is an expression that is built from predicates using logical operators and quantifiers; it may have both free and bound variables.

The two most important quantifiers are the **universal quantifier** $\forall$ and the **existential quantifier** $\exists$:

- The expression '$\forall x \in X, \ldots$' denotes 'for all $x \in X, \ldots$';
- The expression '$\exists x \in X, \ldots$' denotes 'there exists $x \in X$ such that $\ldots$'.

Proving universally quantified logical formulae

When X is finite, we can prove that a property $p(x)$ is true of all the elements $x \in X$ just by checking them one by one. But what if X is infinite?

Example 3
Prove that the square of every odd integer is odd.

The key to Example 3 was introducing a new variable n that refers to an odd integer and, without assuming anything about n other than that it is an odd integer, proving that n^2 is even. We say that n is an *arbitrary* odd integer.

A proof of $\forall x \in X,\ p(x)$ typically looks a bit like this:

- Introduce a variable x, which refers to an element of X.
- Prove $p(x)$, assuming nothing about x except that it is an element of X.

Useful phrases for introducing an arbitrary variable include 'fix $x \in X$' or 'let $x \in X$' or 'take $x \in X$'.

Example 4

Prove that every integer is rational.

Example 5

Prove that, for all irrational numbers x and y, the numbers $x + y$ and $x - y$ are not both rational.

Proving existentially quantified logical formulae

In order to prove that an element of a set X satisfying a property $p(x)$ exists, all we need to do is find one! (Well, and prove that $p(x)$ truly does hold of that element.)

Example 6
Prove that there is a natural number that is a perfect square and is one more than a perfect cube.

The following exercise involves both a universal and an existential quantifier.

Example 7
Prove that, for all $x, y \in \mathbb{Q}$, if $x < y$ then there is some $z \in \mathbb{Q}$ with $x < z < y$.

Uniqueness

Sometimes we want to know not just that an object with a certain property *exists*, but that there is *exactly one* of them. This property is called *uniqueness*. We write $\exists! x \in X,\, p(x)$ to mean that there is exactly one $x \in X$ making $p(x)$ true.

Proving that there is one and only one element x of a set X making a property true is typically done in two stages:

- (**Existence**) Prove that *at least* one $x \in X$ makes $p(x)$ true:

$$\exists x \in X,\, p(x)$$

- (**Uniqueness**) Prove that *at most* one $x \in X$ makes $p(x)$ true:

$$\forall a, b \in X,\, [p(a) \wedge p(b) \Rightarrow a = b] \qquad \text{or} \qquad \underbrace{\forall y \in X,\, [p(y) \Rightarrow y = x]}_{\text{relative to the } x \text{ we proved } \textit{exists}}$$

Example 8
Prove that for all $a \in \mathbb{R}$, there is a unique $x \in \mathbb{R}$ such that $x^2 + 2ax + a^2 = 0$.

Pre-class assignment for Class 5 (Mon, Jan 14)

Read §1.3 *Logical equivalence* up to and including Example 1.3.3, and then answer the questions on Canvas (go to Assignments → Class 5).

Strategies for proving statements involving quantifiers

Strategy (Proving universally quantified statements)
To prove a proposition of the form $\forall x \in X,\ p(x)$, it suffices to prove $p(x)$ for an **arbitrary** element $x \in X$—in other words, prove $p(x)$ whilst assuming nothing about the variable x other than that it is an element of X. $\lhd$

Strategy (Proving existentially quantified statements)
To prove a proposition of the form $\exists x \in X,\ p(x)$, it suffices to prove $p(a)$ for some **specific** element $a \in X$, which should be explicitly defined. $\lhd$

Strategy (Proving unique-existentially quantified statements)
A proof of a statement of the form $\exists ! x \in X,\ p(x)$, consists of two parts:

- **Existence** — prove that $\exists x \in X,\ p(x)$ is true;
- **Uniqueness** — let $a, b \in X$, assume that $p(a)$ and $p(b)$ are true, and derive $a = b$.

Alternatively, prove existence to obtain a fixed $a \in X$ such that $p(a)$ is true, and then prove $\forall x \in X,\ [p(x) \Rightarrow x = a]$. $\lhd$

Strategies for using statements involving quantifiers as assumptions

Strategy (Assuming universally quantified statements)
If an assumption in a proof has the form $\forall x \in X,\ p(x)$, then we may assume that $p(a)$ is true whenever a is an element of X. $\lhd$

Strategy (Assuming existentially quantified statements)
If an assumption in the proof has the form $\exists x \in X,\ p(x)$, then we may introduce a new variable $a \in X$ and assume that $p(a)$ is true. $\lhd$