Math 300 Class 3

Wednesday 9th January 2019

Propositional formulae and logical operators

Definition 1
A **propositional formula** is an expression that is built out of **propositional variables** $p, q, r, \ldots$ using **logical operators** (to be defined soon).

The main logical operators of interest to us are:

- **Conjunction** ($\wedge$), where $p \wedge q$ represents 'p and q';
- **Disjunction** ($\vee$), where $p \vee q$ represents 'p or q;
- **Implication** ($\Rightarrow$), where $p \Rightarrow q$ represents 'if p, then q';
- **Negation** ($\neg$), where $\neg p$ represents 'not p'.

A fifth logical operator, *biimplication* ($\Leftrightarrow$), is defined in terms of $\Rightarrow$ and $\wedge$ by letting $p \Leftrightarrow q$ be shorthand for $(p \Rightarrow q) \wedge (q \Rightarrow p)$; we read $p \Leftrightarrow q$ as 'p if and only if q'.

Exercise 2
Fix integers a and b. Let p represent '$a+b$ is even', let q represent 'a is even', and let r represent 'b is odd'. Consider the following logical formula:

$$p \Leftrightarrow ((q \wedge r) \vee ((\neg q) \wedge (\neg r))$$

Translate it into plain English.

The precise definitions of $\wedge$, $\vee$, $\Rightarrow$ and $\neg$ are *important* (and in the book), but the most useful thing for us is the *proof strategies* that they suggest. In this class we focus on a handful of examples—a more complete list is on page 5

Proving implications

You will often find that a goal in your proof is of the form $p \Rightarrow q$. In this case, the goal can be proved by *assuming* p and *deriving* q.

Exercise 3
Let $m, n \in \mathbb{N}$. Prove that if $2m+1 = 2n+1$, then $m = n$.

Proving biimplications

The biimplication $p \Leftrightarrow q$ asserts that p and q are somehow *equivalent*. A typical proof consists of two steps: first prove $p \Rightarrow q$, and then prove $q \Rightarrow p$. The proposition $q \Rightarrow p$ is called the **converse** of $p \Rightarrow q$.

Exercise 4
Let $x \in \mathbb{R}$. Prove that $x \in \mathbb{Q}$ if and only if $kx \in \mathbb{Z}$ for some nonzero $k \in \mathbb{Z}$.

Proof by cases

When an *assumption* in a proof takes the form $p \vee q$, this means that we know that at least one of p or q is true, but we don't necessarily know which. As such, we need to split into cases based on whether p is true or q is true, and complete the proof in both cases.

Exercise 5
Prove that every positive proper divisor of 9 is odd.

A useful example of when you might use proof by cases is the *law of excluded middle*:

Axiom 6 — *Law of excluded middle*
Let p be any proposition. Then $p \vee (\neg p)$ is true.

Exercise 7
Let x and y be irrational numbers. Prove that x^y may be rational.

Proof by contradiction

We can only prove that things are *true*, but saying that p is false is exactly the same as saying that $\neg p$ is true. Proofs of negations are called proofs *by contradiction*.

Definition 8
A **contradiction** is a proposition that is known or assumed to be false.

In order to prove a goal of the form $\neg p$, we assume p is true and derive a contradiction.

Exercise 9
Let $x \in \mathbb{Q}$. Prove that $\sqrt{2}+x$ is irrational.

Pre-class assignment for Class 4 (Fri, Jan 11)

Read §1.2 *Variables and quantification* up to (but excluding) Definition 1.2.9, and then answer the questions on Canvas (go to Assignments → Class 4).

You do not need to complete Exercises 1.2.6 and 1.2.8.

Strategies for proving propositional formulae

Strategy (Proving conjunctions)
A proof of the proposition $p \wedge q$ can be obtained by tying together two proofs, one being a proof that p is true and one being a proof that q is true. ◁

Strategy (Proving disjunctions)
In order to prove a proposition of the form $p \vee q$, it suffices to prove just one of p or q. ◁

Strategy (Proving implications)
In order to prove a proposition of the form $p \Rightarrow q$, it suffices to assume that p is true, and then derive q from that assumption. ◁

Strategy (Proving negations—proof by contradiction)
In order to prove a proposition p is false (that is, that $\neg p$ is true), it suffices to assume that p is true and derive a contradiction. ◁

Strategies for using propositional formulae as assumptions

Strategy (Assuming disjunctions)
If an assumption in a proof has the form $p \wedge q$, then we may assume p and assume q in the proof. ◁

Strategy (Assuming disjunctions—proof by cases)
If an assumption in a proof has the form $p \vee q$, then we may derive a proposition r by splitting into two cases: first, derive r from the temporary assumption that p is true, and then derive r from the temporary assumption that q is true. ◁

Strategy (Assuming implications—modus ponens)
If an assumption in a proof has the form $p \Rightarrow q$, and p is also assumed to be true, then we may also assume that q is true. ◁

Strategy (Assuming negations)
If an assumption in a proof has the form $\neg p$, then any derivation of p leads to a contradiction. ◁

The law of excluded middle

Strategy (Using the law of excluded middle)
In order to prove a proposition q is true, it suffices to split into cases based on whether some other proposition p is true or false, and prove that q is true in each case. ◁