Math 300 Class 2

Tuesday 8th January 2019

Recall from yesterday that the sets of natural numbers, integers, rational numbers and real numbers are respectively denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. Today we focus on the rational numbers, whose definition we repeat now.

Definition 1 — *Rational and irrational numbers*
A **rational number** is a real number x with the property that there exist $a, b \in \mathbb{Z}$ with $b \neq 0$ such that $x = \frac{a}{b}$. An **irrational number** is a real number that is not rational.

The set of all irrational numbers doesn't have a fancy single-letter name, but after we've studied set operations, we'll be able to write it as '$\mathbb{R} \setminus \mathbb{Q}$'.

Exercise 2
Show that if $x, y \in \mathbb{Q}$, then $xy \in \mathbb{Q}$ and $x + y \in \mathbb{Q}$.

Let $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$ s.t. $x = \frac{a}{b}$ and $y = \frac{c}{d}$ — these exist by Def 1.

Then $xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

and $x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{ad}$

Since $bd \in \mathbb{Z}$ and $bd \neq 0$ we have:

- $ac \in \mathbb{Z} \Rightarrow xy \in \mathbb{Q}$;
- $ad + bc \in \mathbb{Z} \Rightarrow x + y \in \mathbb{Q}$,

as required.

Theorem 3
The real number $\sqrt{2}$ is irrational. $\square$

The proof that $\sqrt{2}$ is irrational has a classic *proof by contradiction*—this is a proof technique that we will see soon, and we will postpone the proof of Theorem 3 until then.

Exercise 4
Let x and y be irrational numbers. Prove that it is possible that xy be rational.

Let $x = y = \sqrt{2}$. Then x and y are irrational and $xy = \sqrt{2} \cdot \sqrt{2} = 2 = \frac{2}{1} \Rightarrow xy \in \mathbb{Q}$.

Exercise 5
Let x be a rational number and y be an irrational number. Prove that it is possible that xy be rational, and it is possible that xy be irrational.

- Let $x = 0$ and $y = \sqrt{2}$. Then $x = \frac{0}{1} \Rightarrow x \in \mathbb{Q}$ and y is irrational, so $xy = 0 \cdot \sqrt{2} = 0 \in \mathbb{Q}$.
- Let $x = 1$ and $y = \sqrt{2}$. Then $x = \frac{1}{1} \Rightarrow x \in \mathbb{Q}$ and y is irrational, so $xy = 1 \cdot \sqrt{2} = \sqrt{2}$, which is irrational.

$\square$

Section 1.1. Propositional logic

At any point in a proof, we have a set of **assumptions** and a set of **goals**.

- **Assumptions** are propositions we already know are true, and propositions that we are assuming as hypotheses for the sake of exploring their consequences.
- **Goals** are propositions that we need to prove in order for the proof to be complete.

At the begining of a proof of a proposition P, there are no assumptions (beyond results we already know are true) and there is just one goal, namely P. The proof ends when there are no goals remaining.

The words we use in writing (or speaking) a proof are what change or clarify the assumptions and goals in the proof.

Exercise 6

What follows is the statement and proof of a theorem, with some numbers in [brackets] added for reference between sentences and before and after the proof.

> **Theorem**
> For all $a, b, c \in \mathbb{Z}$, if c divides b and b divides a, then c divides a.

Proof. [1] Let $a, b, c \in \mathbb{Z}$. [2] Suppose c divides b and b divides a. [3] Then $b = qc$ for some $q \in \mathbb{Z}$ and $a = rb$ for some $r \in \mathbb{Z}$. [4] Therefore $a = rb = r(qc) = (rq)c$. [5] Hence c divides a by the definition of divisibility. [6] □

Discuss with a partner what the assumptions and goals are at each of the six stages in the proof.

A framework for writing down your answers to Exercise 6
can be found on the reverse side of this page.

Pre-class assignment for Class 3 (Wed, Jan 9)

Read §1.1 *Propositional logic* up to (but excluding) Definition 1.1.4, and then answer the questions on Canvas (go to Assignments → Class 3).

[1] **Assumptions:** ——

Goals: For all $a, b, c \in \mathbb{Z}$, if c divides b and b divides a, then c divides a.

[2] **Assumptions:** $a, b, c \in \mathbb{Z}$

Goals: If c divides b and b divides a, then c divides a.

[3] **Assumptions:** $a, b, c \in \mathbb{Z}$, c divides b, b divides a

Goals: c divides a $(\equiv \ a = [\text{some integer}] \cdot c)$

[4] **Assumptions:** $a, b, c, q, r \in \mathbb{Z}$, $b = qc$, $a = rb$

Goals: $a = [\text{some integer}] \cdot c$

[5] **Assumptions:** $a, b, c, q, r \in \mathbb{Z}$, $b = qc$, $a = rb$, $a = (rq)c$

Goals: ——

[6] Assumptions: } Same as [5].

Goals }