Math 290-2 Class 26

Monday 11th March 2019

Constrained extrema: one constraint

There is often a need to maximise or minimise a quantity subject to an equational constraint.

Suppose we want to maximise a quantity $f(x,y)$ subject to the constraint $g(x,y) = c$, where c is some constant.

If k is the largest value attained by $f(x,y)$, then the level curve $f(x,y) = k$ must be tangent to the curve $g(x,y) = c$.

(See accompanying illustration.)

This means that the gradient vector to the curve $f(x,y) = k$ must be parallel to the gradient vector to the curve $g(x,y) = c$. Thus at the point (x,y), we have

$$\boxed{\nabla f(x,y) = \lambda \nabla g(x,y)}$$

The scalar λ is called a **Lagrange multiplier**.

The system of equations given by $g(x,y) = c$ and $\nabla f(x,y) = \lambda \nabla g(x,y)$ can be solved, and whichever solution yields the greatest value of $f(x,y)$ is the maximum value of $f(x,y)$ subject to the constraint $g(x,y) = c$. (Likewise, the least value of $f(x,y)$ is the minimum value of $f(x,y)$ subject to the constraint $g(x,y) = c$.)

The points where f attains these maximum and minimum values are called **constrained extrema**.

This generalises to higher dimensions: to maximise (or minimise) $f(\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = c$, solve the system given by $g(\mathbf{x}) = c$ and $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$ and take whichever solution makes the value of $f(\mathbf{x})$ greatest (or least).

Constrained extrema: multiple constraints

Introducing more constraints leads to a system $\mathbf{g}(\mathbf{x}) = \mathbf{c}$; that is

$$g_1(\mathbf{x}) = c_1, \quad g_2(\mathbf{x}) = c_2, \quad \ldots, \quad g_m(\mathbf{x}) = c_m$$

In this case, we need m Lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$, and the system we need to solve is

$$\nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \quad \text{or equivalently} \quad \nabla f(\mathbf{x}) = \lambda_1 \nabla g_1(\mathbf{x}) + \cdots + \lambda_m \nabla g_m(\mathbf{x})$$

1. [*Colley, §4.3 Q36*] Find the maximum value of $\sin\alpha\sin\beta\sin\gamma$, where α, β and γ are the interior angles of a triangle.

2. Let Ω be the plane in $\mathbb{R}^3$ described by the equation $2x - z = 2$. Find the point P on Ω that is closest to the point $Q(4, -1, 1)\ldots$

(a) …by using Lagrange multipliers;

(b) …using least squares approximation;

(c) …by reasoning geometrically.

3. Find a 2×2 symmetric matrix A of greatest determinant such that $\left\| A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = 2$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A with eigenvalue 1.