Math 290-2 Class 25

Friday 8th March 2019

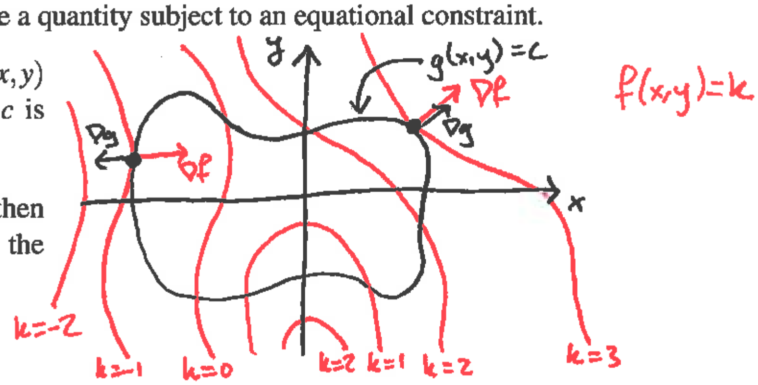
Constrained extrema: one constraint

There is often a need to maximise or minimise a quantity subject to an equational constraint.

Suppose we want to maximise a quantity $f(x,y)$ subject to the constraint $g(x,y) = c$, where c is some constant.

If k is the largest value attained by $f(x,y)$, then the level curve $f(x,y) = k$ must be tangent to the curve $g(x,y) = c$.

(See accompanying illustration.)



This means that the ~~gradient~~ normal vector to the curve $f(x,y) = k$ must be parallel to the ~~gradient~~ normal vector to the curve $g(x,y) = c$. Thus at the point (x,y), we have

$$\boxed{\nabla f(x,y) = \lambda \nabla g(x,y)}$$

The scalar λ is called a **Lagrange multiplier**.

The system of equations given by $g(x,y) = c$ and $\nabla f(x,y) = \lambda \nabla g(x,y)$ can be solved, and whichever solution yields the greatest value of $f(x,y)$ is the maximum value of $f(x,y)$ subject to the constraint $g(x,y) = c$. (Likewise, the least value of $f(x,y)$ is the minimum value of $f(x,y)$ subject to the constraint $g(x,y) = c$.)

The points where f attains these maximum and minimum values are called **constrained extrema**.

This generalises to higher dimensions: to maximise (or minimise) $f(\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = c$, solve the system given by $g(\mathbf{x}) = c$ and $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$ and take whichever solution makes the value of $f(\mathbf{x})$ greatest (or least).

Constrained extrema: multiple constraints

Introducing more constraints leads to a system $\mathbf{g}(\mathbf{x}) = \mathbf{c}$; that is

$$g_1(\mathbf{x}) = c_1, \quad g_2(\mathbf{x}) = c_2, \quad \dots, \quad g_m(\mathbf{x}) = c_m$$

In this case, we need m Lagrange multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$, and the system we need to solve is

$$\nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \quad \text{or equivalently} \quad \nabla f(\mathbf{x}) = \lambda_1 \nabla g_1(\mathbf{x}) + \cdots + \lambda_m \nabla g_m(\mathbf{x})$$

1. Find the points on the ellipse $\boxed{x^2+xy+y^2=3}$ that are closest to the origin.

The distance is minimised when the square of the distance is minimised, so we'll solve the following:

Minimise $\underbrace{x^2+y^2}_{f(x,y)}$ subject to $\underbrace{x^2+xy+y^2}_{g(x,y)}=3$

$$Df=\lambda Dg \iff \begin{pmatrix}2x\\2y\end{pmatrix}=\lambda\begin{pmatrix}2x+y\\x+2y\end{pmatrix}$$

$$\iff \begin{cases}(2-2\lambda)x-\lambda y=0\\-\lambda x+(2-2\lambda)y=0\end{cases} \iff (x,y)=(0,0) \leftarrow \text{impossible — not on the ellipse}$$

$$\text{or } (x,y) \text{ is in } \ker\begin{pmatrix}2-2\lambda & -\lambda\\-\lambda & 2-2\lambda\end{pmatrix}$$

So we need $\underbrace{\det\begin{pmatrix}2-2\lambda & -\lambda\\-\lambda & 2-2\lambda\end{pmatrix}}=0$

$$= (2-2\lambda)^2-\lambda^2 = 3\lambda^2-8\lambda+4$$
$$=(3\lambda-2)(\lambda-2) \Rightarrow \lambda=\tfrac{2}{3} \text{ or } 2$$

<u>If $\lambda=\frac{2}{3}$</u> : $\frac{2}{3}x-\frac{2}{3}y=0 \Rightarrow x=y$

$$\Rightarrow x^2+x\cdot x+x^2=3x^2=3 \Rightarrow x=\pm1=y$$
$$\& \ f(x,y)=1^2+1^2=\underline{\underline{2}}$$

<u>If $\lambda=2$</u> : $-2x-2y=0 \Rightarrow x=-y$

$$\Rightarrow x^2+x\cdot(-x)+(-x)^2=x^2=3 \Rightarrow x=\pm\sqrt{3},\ y=\mp\sqrt{3}$$
$$\& \ f(x,y)=(\pm\sqrt{3})^2+(\mp\sqrt{3})^2$$
$$=\underline{\underline{6}}$$

So the points closest to the origin are $(1,1)$ and $(-1,-1)$.

2. Find the greatest volume that an item of luggage of the largest permissible size can have when flying with American Airlines.

According to the AA website, checked luggage must measure ≤ 62 linear inches, i.e. $x + y + z \leq 62$ where x = width, y = length, z = height (in inches).

The volume will be maximised when $x + y + z = 62$ (otherwise increasing x, y or z will increase xyz).

$g(x,y,z)$

So we need to maximise xyz subject to $x+y+z=62$ (and $x, y, z \geq 0$).

$f(x,y,z)$

$$\nabla f = \lambda \nabla g \iff \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$xyz = 0$

$$\begin{cases} yz - \lambda = 0 \\ xz - \lambda = 0 \\ xy - \lambda = 0 \end{cases} \Rightarrow xz = yz \Rightarrow z = 0 \text{ or } x = y$$

$xyz = 0$

<u>If $x = y$</u>: $\begin{cases} xz - \lambda = 0 \\ x^2 - \lambda = 0 \end{cases} \Rightarrow xz = x^2 \Rightarrow x = 0$ or $x = z$

<u>If $x = z$</u>: $x + y + z = 3x = 62 \Rightarrow x = \frac{62}{3}$

$$\Rightarrow xyz = \left(\frac{62}{3}\right)^3 = \frac{238328}{27} \approx 8827 \text{ in}^3 \approx 5.1 \text{ ft}^3$$

So the maximum possible volume is ≈ 5.1 ft^3 when the luggage has width, length & height equal to $\frac{62}{3}$ (≈ 20.7) inches.

3. Find the point on the line of intersection of the planes $x+2y-z=1$ and $2x-z=3$ that is closest the point $(1,0,-1)$

Again we minimise the square of the distance:

Minimise $\underbrace{(x-1)^2+y^2+(z+1)^2}_{f(x,y,z)}$ subject to $\begin{cases} x+2y-z=1 & \leftarrow g_1 \\ 2x \quad\;\; -z=3 & \leftarrow g_2 \end{cases}$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \iff \begin{pmatrix} 2x-2 \\ 2y \\ 2z+2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

This + the constraints give the following linear system:

$$\begin{cases} x + 2y - z & = 1 \\ 2x \quad\;\; - z & = 3 \\ 2x \qquad\qquad - \lambda_1 - 2\lambda_2 & = 2 \\ \quad\; 2y \qquad - 2\lambda_1 & = 0 \\ \qquad\quad 2z + \lambda_1 + \lambda_2 & = -2 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 0 & 1 \\ 2 & 0 & -1 & 0 & 0 & 3 \\ 2 & 0 & 0 & -1 & -2 & 2 \\ 0 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -2 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 22/21 \\ 0 & 1 & 0 & 0 & 0 & -10/21 \\ 0 & 0 & 1 & 0 & 0 & -19/21 \\ 0 & 0 & 0 & 1 & 0 & -10/21 \\ 0 & 0 & 0 & 0 & 1 & 6/21 \end{array}\right)$$

So $(x,y,z) = \frac{1}{21}(22, -10, -19)$ is the unique solution

(and then the distance is $\sqrt{5/21}$.)

It is a minimum, not a maximum, since e.g. $(0,-1,-3)$ is on the line of intersection of the planes & its distance from $(1,0,-1)$ is $\sqrt{1^2+1^2+2^2}$

$$= \sqrt{6} > \sqrt{5/21}$$