

Math 290-2 Class 25

Friday 8th March 2019

Constrained extrema: one constraint

There is often a need to maximise or minimise a quantity subject to an equational constraint.

Suppose we want to maximise a quantity $f(x,y)$ subject to the constraint $g(x,y) = c$, where c is some constant.

If k is the largest value attained by $f(x,y)$, then the level curve $f(x,y) = k$ must be tangent to the curve $g(x,y) = c$.

(See accompanying illustration.)

This means that the gradient vector to the curve $f(x,y) = k$ must be parallel to the gradient vector to the curve $g(x,y) = c$. Thus at the point (x,y) , we have

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

The scalar λ is called a **Lagrange multiplier**.

The system of equations given by $g(x,y) = c$ and $\nabla f(x,y) = \lambda \nabla g(x,y)$ can be solved, and whichever solution yields the greatest value of $f(x,y)$ is the maximum value of $f(x,y)$ subject to the constraint $g(x,y) = c$. (Likewise, the least value of $f(x,y)$ is the minimum value of $f(x,y)$ subject to the constraint $g(x,y) = c$.)

The points where f attains these maximum and minimum values are called **constrained extrema**.

This generalises to higher dimensions: to maximise (or minimise) $f(\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = c$, solve the system given by $g(\mathbf{x}) = c$ and $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$ and take whichever solution makes the value of $f(\mathbf{x})$ greatest (or least).

Constrained extrema: multiple constraints

Introducing more constraints leads to a system $\mathbf{g}(\mathbf{x}) = \mathbf{c}$; that is

$$g_1(\mathbf{x}) = c_1, \quad g_2(\mathbf{x}) = c_2, \quad \dots, \quad g_m(\mathbf{x}) = c_m$$

In this case, we need m Lagrange multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$, and the system we need to solve is

$$\nabla f(\mathbf{x}) = \boldsymbol{\lambda}^T D\mathbf{g}(\mathbf{x}) \quad \text{or equivalently} \quad \nabla f(\mathbf{x}) = \lambda_1 \nabla g_1(\mathbf{x}) + \dots + \lambda_m \nabla g_m(\mathbf{x})$$

1. Find the points on the ellipse $x^2 + xy + y^2 = 3$ that are closest to the origin.

2. Find the greatest volume that an item of luggage of the largest permissible size can have when flying with American Airlines.

3. Find the point on the line of intersection of the planes $x + 2y - z = 1$ and $2x - z = 3$ that is closest the point $(1, 0, -1)$