Math 290-2 Class 24

Wednesday 6th March 2019

Global extrema

A **global maximum** of a function $f : \mathbb{R}^n \to \mathbb{R}$ within a region D of $\mathbb{R}^n$ is a point $\mathbf{a}$ in D such that $f(\mathbf{a}) \geqslant f(\mathbf{x})$ for all points $\mathbf{x}$ in D. A **global minimum** is defined likewise, and a **global extremum** is a point that is either a global maximum or a global minimum.

In general, a function might attain no global maximum or minimum value. However, the **extreme value theorem** tells us that if D is a region of $\mathbb{R}^n$ that is...

- ... **closed** (it contains all of its boundary points); and
- ... **bounded** (there is an upper bound on how far apart two points of D can be);

... then f attains both a maximum and minimum value in D—that is, there are points $\mathbf{a}_{\text{max}}$ and $\mathbf{a}_{\text{min}}$ in D such that

$$f(\mathbf{a}_{\text{min}}) \leqslant f(\mathbf{x}) \leqslant f(\mathbf{a}_{\text{max}})$$

for all $\mathbf{x}$ in D. (The fancy name for a closed and bounded region of $\mathbb{R}^n$ is a *compact set*.)

In order to find the global extrema of f on a compact set D:

- Find the critical points of f inside D.
- Find the global extrema of f on the boundary of D.

  'Officially', what you need to do here is find a parametrisation $\mathbf{x}(\mathbf{t})$ of the boundary of D—this gives rise to a new function $g(\mathbf{t}) = f(\mathbf{x}(\mathbf{t}))$, where $\mathbf{t}$ ranges over some suitable region of $\mathbb{R}^{n-1}$, whose global extrema you can find by repeating this method.

  This sounds scary, but it really isn't—it is best illustrated by example.
- Amongst the points that you found, a point where the value of f is least is a global minimum of f on D, and a point where the value of f is greatest is a global maximum.

If you want to find the global extrema of f on all of $\mathbb{R}^n$, then you should find its local extrema and make sure that the function truly does attain a local minimum and local maximum value.

1. Find the global extrema of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 - 2xy - y^2$ on the closed circular disc of radius 1 centred at $(0, 0)$.

2. Find the maximum and minimum values attained by the function $f(x,y) = \dfrac{x^2 - y^2}{1 + x^2 + y^2}$ on the semicircular region of $\mathbb{R}^2$ defined in polar coordinates by $0 \leqslant r \leqslant 1$, $\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{5\pi}{4}$.

3. For each of the following statements, determine whether it is true or false.

(a) If a function $f : \mathbb{R}^n \to \mathbb{R}$ has a unique critical point $\mathbf{a}$, and $\mathbf{a}$ is a local minimum of f, then $\mathbf{a}$ is a global minimum of f.

(b) If D is a bounded region of $\mathbb{R}^n$, and f is a bounded function on X (that is, there are real numbers a and b such that $a \leqslant f(\mathbf{x}) \leqslant b$ for all $\mathbf{x}$ in D), then f has a global minimum and a global maximum on D.

(c) If D is a closed region of $\mathbb{R}^n$, and f is a function on X that is unbounded above (that is, for any $a > 0$, there is some $\mathbf{x}$ in D such that $f(\mathbf{x}) > a$), then D is unbounded.