Math 290-2 Class 23

Monday 4th March 2019

Local extrema, saddle points and critical points

A **local maximum** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a point $\mathbf{a}$ such that $f(\mathbf{a}) \geqslant f(\mathbf{x})$ for all $\mathbf{x}$ in some neighbourhood of $\mathbf{a}$. A **local minimum** is defined similarly, and a **local extremum** is a point that is either a local maximum or a local minimum.

It might be the case that $f(\mathbf{a})$ is a local maximum value of $f(\mathbf{x})$ as $\mathbf{x}$ moves in one direction, but a local minimum value of $f(\mathbf{x})$ as $\mathbf{x}$ moves in a different direction. If this is the case, we say $\mathbf{a}$ is a **saddle point** of f.

[For example, $(0,0)$ is a local minimum of $f(x,y) = x^2 - y^2$ as (x,y) moves along the line $y = 0$, but a local maximum of as (x,y) moves along the line $x = 0$.]

A **critical point a** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a point where either:

- $f(\mathbf{a})$ is defined but f is not differentiable at $\mathbf{a}$; or
- f is differentiable at $\mathbf{a}$ and $\nabla f(\mathbf{a}) = \mathbf{0}$.

Local extrema and saddle points always occur at critical points. So to find the local extrema of a function, it suffices to find its critical points and examine the behaviour of the function around those points.

Classifying critical points

If f is twice differentiable at $\mathbf{a}$, then f is approximated by its second-order Taylor polynomial at $\mathbf{a}$. Therefore, if $\nabla f(\mathbf{a}) = \mathbf{0}$, then

$$f(\mathbf{x}) \approx \underbrace{f(\mathbf{a}) + \frac{1}{2}(\mathbf{x}-\mathbf{a})^T Hf(\mathbf{a})(\mathbf{x}-\mathbf{a})}_{=\,Q(\mathbf{x}),\text{ since }\nabla f(\mathbf{a})=\mathbf{0}} \quad \text{when } \mathbf{x} \approx \mathbf{a}$$

This means that, near $\mathbf{a}$, the graph of f 'looks like' the graph of the quadratic form corresponding to the (symmetric!) matrix $Hf(\mathbf{a})$. Therefore:

- If the eigenvalues of $Hf(\mathbf{a})$ are all positive, then $\mathbf{a}$ is a local minimum;
- If the eigenvalues of $Hf(\mathbf{a})$ are all negative, then $\mathbf{a}$ is a local maximum;
- If $Hf(\mathbf{a})$ has some positive and some negative eigenvalues, then $\mathbf{a}$ is a saddle point.

If f is not twice differentiable at $\mathbf{a}$, or any of the eigenvalues of $Hf(\mathbf{a})$ are zero, then we need to look a bit more closely.

1. [*Colley, §4.2 Q3*] Find and classify the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = 2xy - 2x^2 - 5y^2 + 4y - 3$$

2. Find and classify the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \cos(x)\cos(y)$ lying in the region $-\pi < x < \pi$, $-\pi < y < \pi$.

3. [*Colley, §4.2 Q19*] Find and classify the critical points of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y,z) = xy + xz + 2yz + \frac{1}{x}$$